

# The Plateau Problem for General Curvature Functions

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**Abstract:** We use a novel, differential topological approach to solve the Plateau problem for general convex curvature functions in general Hadamard manifolds.

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# 1 - Introduction.

## 1.1 Convex Curvature Functions.

In this paper we prove the existence of solutions to the Plateau Problem in Hadamard manifolds for locally strictly convex (LSC in the sequel) hypersurfaces of constant curvature for a large class of curvature functions. We first describe the curvature functions used, using [5] as our guide. For  $n \in \mathbb{N}$ , let  $\Gamma^n \subseteq \mathbb{R}^n$  be the open cone of vectors all of whose components are strictly positive. Let  $f \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$  be a non-negative valued function such that:

**Axiom (i):** for every permutation,  $\sigma$ , and for all  $x_1, \dots, x_n \in \Gamma$ :

$$f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n).$$

We say that  $f$  is a **convex curvature function** if and only if, in addition to satisfying Axiom (i):

**Axiom (ii):**  $f$  is homogeneous of order 1;

**Axiom (iii):**  $f(1, 1, \dots, 1) = 1$ ; and

**Axiom (iv):**  $f$  is strictly positive over  $\Gamma$  and vanishes over  $\partial\Gamma$ .

Scalar notions of curvature of hypersurfaces are generated by convex curvature functions. Indeed, let  $M := M^{n+1}$  be an  $(n+1)$ -dimensional Riemannian manifold. Let  $\Sigma = (S, i)$  be an LSC immersed hypersurface in  $M$  and let  $A$  be the shape operator of  $\Sigma$ . If  $K$  is a convex curvature function, we define  $K_\Sigma$ , the  **$K$ -curvature** of  $\Sigma$ , by:

$$K_\Sigma = K(\lambda_1, \dots, \lambda_n),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ . We now see that Axiom (i) ensures that  $K$ -curvature is well defined; Axiom (ii) ensures that it transforms in a familiar manner under rescalings of the metric over  $M$ ; Axiom (iii) is merely a normalisation condition; and Axiom (iv) ensures that strict convexity is not lost after taking a smooth limit of strictly convex hypersurfaces of strictly positive prescribed  $K$ -curvature.

Let  $f$  be a convex curvature function, we say that  $f$  is **admissible** if and only if:

**Axiom (v):**  $f$  is strictly elliptic. In other words, for all  $x \in \Gamma \subseteq \mathbb{R}^n$  and for all  $1 \leq i \leq n$ :

$$(\partial_i f)(x) > 0;$$

**Axiom (vi):**  $f$  is a concave function over  $\Gamma \subseteq \mathbb{R}^n$ .

These conditions are natural from the point of view of non-linear elliptic partial differential equations (c.f. [4]). Finally, in order to study boundary regularity of hypersurfaces of prescribed  $K$ -curvature, we require, in addition, one supplementary property. If  $f$  is an admissible convex curvature function, we define  $f_\infty \in C^0(\Gamma^{n-1})$  by:

$$f_\infty(x_1, \dots, x_{n-1}) = \lim_{t \rightarrow +\infty} f(x_1, \dots, x_{n-1}, t).$$

By Axiom (v), the quantity inside the limit is a strictly increasing function of  $t$ , and so  $f_\infty$  is well defined. In Proposition 2.1, we see that, over the interior of  $\Gamma^{n-1}$ ,  $f_\infty$  is either everywhere infinite, or everywhere finite, in which case it is also concave, elliptic and homogeneous of order 1. We then say that  $f$  is **regular at infinity** if and only if either:

**Axiom (vii):**  $f_\infty$  is everywhere infinite; or

**Axiom (vii'):**  $f_\infty$  is finite and satisfies the following two conditions:

(a):  $f_\infty$  is strictly elliptic. In other words, for all  $x \in \Gamma^{n-1}$ , there exists a supporting tangent  $(\xi_1, \dots, \xi_{n-1})$  to  $f_\infty$  at  $x$  such that for all  $i$ :

$$\xi_i > 0; \text{ and}$$

(b): for every compact subset  $X \subseteq \Gamma^{n-1}$ , there exists  $C, T > 0$  such that, for all  $x \in X$  and for all  $t \geq T$ :

$$f(x, t) \leq f_\infty(x) - Ct^{-1}.$$

*Remark:* Axioms (i) to (vii) are satisfied by a large class of curvature functions, including Gaussian curvature and special Lagrangian curvature (c.f. [23] and [24]). Of particular interest are also the curvature quotients defined as follows: for all  $0 \leq k \leq n$ , let  $\sigma_k$  be the  $k$ 'th symmetric polynomial of  $x$ . Thus:

$$\sigma_k(x) = \sum_{|I|=k} x_{i_1} \cdot \dots \cdot x_{i_k}.$$

For  $1 \leq k < n$ , we define  $f_{n,k}$  by:

$$f_{n,k} = c_{n,k}(\sigma_n/\sigma_k)^{1/(n-k)},$$

where:

$$c_{n,k} = (\sigma_k(1))^{1/(n-k)} = \binom{n}{k}^{1/(n-k)}.$$

In Proposition 2.2, we see that these curvature quotients all satisfy Axioms (i) to (vi) and also (vii'), and therefore fall within the scope of this paper.  $\square$

## 1.2 The Plateau Problem with Outer and Inner Barrier.

We begin by studying the Plateau problem in the slightly simpler case where there are two barriers corresponding to a subsolution and a supersolution, which generalises the existence results [5] of Caffarelli, Nirenberg and Spruck and [7] of Guan. We formulate the problem as follows: let  $M := M^{n+1}$  be a Hadamard manifold. Let  $\Sigma_l := (\Sigma_l, \partial\Sigma_l)$  and  $\Sigma_u := (\Sigma_u, \partial\Sigma_u)$  be smooth, compact, LSC, isometrically immersed hypersurfaces in  $M$  with smooth boundary. We say that  $\Sigma_u$  **bounds**  $\Sigma_l$ , and we denote  $\Sigma_u > \Sigma_l$  if and only if there exists a smooth, compact  $(n+1)$ -dimensional manifold  $(N, \partial N)$  with piecewise smooth boundary and a smooth isometric immersion  $I : N \rightarrow M$  such that  $\partial N$  consists of 2 connected components  $\partial N_1$  and  $\partial N_2$  and:

- (i)  $\partial N_1$  and  $\partial N_2$  coincide and  $\Gamma := \partial N_1 = \partial N_2$  is smooth;
- (ii)  $(\partial N_1, I) = \Sigma_l$  and  $N$  lies on the outside of this hypersurface;
- (iii)  $(\partial N_2, I) = \Sigma_u$  and  $N$  lies on the inside of this hypersurface; and
- (iv)  $N$  is foliated by the geodesic segments normal to  $\partial N_1$ .

*Remark:* Importantly, if  $\Sigma_u$  is a graph over  $\Sigma_l$ , then  $\Sigma_u$  trivially bounds  $\Sigma_l$ . Likewise, if  $\Sigma_u$  bounds  $\Sigma_l$  and these two surfaces make an angle of less than  $\pi/2$  at every point of  $\Gamma := \partial \Sigma_u = \partial \Sigma_l$ , then  $\Sigma_u$  is a graph over  $\Sigma_l$ . We thus see that the concept of bounding cleanly generalises the concept of graph for LSC immersions.  $\square$

This notion is closely related to that of cobordism and we will see that it indeed defines a partial order on the family of LSC immersed hypersurfaces, justifying the notation (see Proposition 3.6).  $N$  is trivially unique up to isometric reparametrisation, and in the sequel, we will refer to  $N := (I, (N, \partial N))$  as the **convex cobordism** from  $\Sigma_l$  to  $\Sigma_u$ .

Let  $DK$  denote the linearisation of the  $K$ -curvature operator, which, we recall, is a generalised Laplacian acting on smooth functions over  $S$ . Now consider any LSC immersed hypersurface  $\Sigma := (\Sigma, \partial \Sigma)$  in  $M$  and let  $\mathbf{N}$  be the outward pointing unit normal vector field over  $\Sigma$ . For  $\kappa \in C^\infty(M)$ , define the operator  $\mathcal{L}_\kappa$  on functions over  $S$  by:

$$\mathcal{L}_\kappa f = DKf - \langle \nabla \kappa, \mathbf{N} \rangle f.$$

We say that  $\Sigma$  is **non-degenerate** for  $(K, \kappa)$  if and only if  $\mathcal{L}_\kappa$  is invertible, and we say that it is **stable** for  $(K, \kappa)$  if and only if, in addition, for all non-negative  $f \in C^\infty(S)$ , the unique solution,  $g$ , to  $\mathcal{L}_\kappa g = f$  is also non-negative.

We now obtain:

### Theorem 1.1

Let  $K$  be an admissible convex curvature function which is regular at infinity; let  $M$  be a Hadamard manifold of sectional curvature bounded above by  $-1$ ; Let  $\kappa_0 < \kappa \in C^\infty(M)$  be smooth functions taking values in  $]0, 1[$ ; and let  $\Sigma_l := (\Sigma_l, \partial \Sigma_l)$  and  $\Sigma_u := (\Sigma_u, \partial \Sigma_u)$  be smooth, compact, LSC immersed hypersurfaces such that:

- (i)  $\Sigma_u > \Sigma_l$ ;
- (ii)  $K(\Sigma_l) = \kappa_0$ ; and
- (iii)  $K(\Sigma_u) > \kappa$ .

Then, if  $\Sigma_l$  is non-degenerate and stable for  $(K, \kappa_0)$ , there exists a smooth, LSC immersed hypersurface  $\Sigma := (\Sigma, \partial \Sigma)$ , distinct from  $\Sigma_l$  such that:

$$\Sigma_u > \Sigma > \Sigma_l,$$

and one of the following two possibilities holds:

- (a) either  $K(\Sigma) = \kappa_0$ ;
- (b) or  $K(\Sigma) = \kappa$ .

*Remark:* This theorem generalises Theorem 1.1 of [22] to general notions of curvature.  $\square$

*Remark:* The same result holds with  $\Sigma_u$  taking the role of  $\Sigma_l$  in the hypothesis of this theorem.  $\square$

*Remark:* This theorem is essentially a statement about the number of solutions, or degree, counted modulo 2. The non-degeneracy of  $\Sigma_l$  ensures that the degree is well defined, and in case (a) it is equal to 0, whereas in case (b) it is equal to 1.  $\square$

*Remark:* By calculating  $K$  as in Lemma 7.2 of [25], we can show that  $\Sigma_l$  is non-degenerate and stable for  $(K, \kappa_0)$  whenever its shape operator is bounded above by  $\text{Id}$ .  $\square$

### 1.3 The Plateau Problem with Outer Barrier.

We now consider the more general Plateau problem where only an outer barrier is given. This problem was first conjectured by Spruck in [28] and was then solved independently in the case where  $K$  is Gaussian curvature and  $M = \mathbb{R}^{n+1}$  by Guan and Spruck in [8] and Trudinger and Wang in [29]. These results were further developed by Guan and Spruck in [9], by Ivochkina and Tomi in [13], and by Sheng, Urbas and Wang in [21] to treat more general convex curvature functions still in the case where  $M = \mathbb{R}^{n+1}$ .

All these papers use the Perron Method, which, in this context, may only be used when the ambient manifold is affine flat. In the case of special Lagrangian curvature, we were able to use the special properties that this notion of curvature possesses in [24] and [27] to apply the Perron Method in general manifolds. However, special Lagrangian curvature constitutes a very special case, and in general, for most other notions of curvature, with the current technology, the Perron Method breaks down in general manifolds, and a novel technique is required. It is for this reason that we apply the differential topological approach developed by the author in [26], which not only allows us to solve the Plateau Problem in much greater generality, but moreover, via ideas of [12], [15] and [19], allows us to prove uniqueness in certain cases, as will be seen presently.

We make the following definitions: let  $K$  be an admissible convex curvature function which is regular at infinity. We say that  $K$  is of **finite type** if and only if  $K_\infty$  extends to a continuous function over the closure of  $\Gamma^{n-1}$  which vanishes along  $\partial\Gamma^{n-1}$ , and we say that  $K$  is of **determinant type** if and only if:

$$K = \sum_{i=1}^n \gamma_i K_0^{\alpha_i} K_i^{\beta_i},$$

where  $K_0 = \text{Det}^{1/n}$ ,  $\gamma_1 + \dots + \gamma_n = 1$  and, for all  $i$ :

- (i)  $K_i$  is of finite type; and
- (ii)  $\alpha_i + \beta_i = 1$ .

Importantly, admissible convex curvature functions of finite or determinant type constitute a large family which includes the curvature quotients  $f_{n,k}$ , for all  $n, k$ , special Lagrangian curvature, and Gaussian curvature.

We prove:

**Theorem 1.2**

Let  $K$  be an admissible convex curvature function of finite or determinant type; let  $M$  be an  $(n+1)$ -dimensional Hadamard manifold of sectional curvature bounded above by  $-1$ ; let  $\kappa \in C^\infty(M)$  be a smooth function taking values in  $]0, 1[$ ; and let  $\Sigma_u := (\Sigma_u, \partial\Sigma_u)$  be a smooth, compact, LSC immersed hypersurface such that:

- (i)  $K(\Sigma_u) > \kappa$ ; and
- (ii)  $\partial\Sigma_u$  only self-intersects transversally.

Then there exists a smooth, compact, LSC immersed hypersurface  $\Sigma := (\Sigma, \partial\Sigma)$  such that:

- (i)  $\Sigma_u > \Sigma$ ; and
- (ii)  $K(\Sigma) = \kappa$ .

In addition, we obtain uniqueness in certain cases:

**Theorem 1.3**

Moreover, if, for all  $A \in \Gamma$  such that  $K(A) < 1$ :

$$DK_A \cdot \text{Id} > DK_A \cdot A^2,$$

then the solution is unique.

*Remark:* Importantly, this property is satisfied by the curvature quotients  $f_{n,k}$  for  $k \in \{n-1, n-2\}$  (c.f. [12]), by special Lagrangian curvature (c.f. Lemma 7.4 of [25]) and by Gaussian curvature when the hypersurface is 2-dimensional (c.f. Proposition 3.2.1 of [15]).  $\square$

### 1.4 An Alternative Formulation.

The curvature condition on  $M$  is required to obtain global bounds on the norm of the shape operator of an immersed hypersurface of prescribed  $K$ -curvature. These bounds are obtained using the maximum principal by feeding the horosphere foliation of  $M$  into the following lemma which we believe to be of independant interest: let  $K$  be an admissible convex curvature function, let  $\Sigma$  be a smooth, LSC, immersed hypersurface in  $M$  and let  $A$  be the shape operator of  $K$ . By Proposition 2.3,  $DK_A$  defines a symmetric matrix. Choose  $p \in \Sigma$ , and let  $e_1, \dots, e_n \in T\Sigma$  be an orthonormal basis of eigenvectors  $DK_A$  and let  $\mu_1, \dots, \mu_n$  be their corresponding eigenvalues. We define the operator,  $\Delta^K$  over  $\Sigma$  such that, for all  $f \in C^\infty(\Sigma)$ , with respect to this basis:

$$\Delta^K f = \sum_{i=1}^n \mu_i \text{Hess}^\Sigma(f)_{ii},$$

where  $\text{Hess}^\Sigma$  is the Hessian of the Levi-Civita covariant derivative of  $\Sigma$ .

**Lemma 6.2**

Let  $K$  be an admissible convex curvature function; let  $\kappa : M \rightarrow ]0, \infty[$  be a smooth strictly positive function; let  $\Sigma$  be an LSC smooth, immersed hypersurface in  $M$  such that  $K(\Sigma) = \kappa$ ; and let  $\varphi : M \rightarrow \mathbb{R}$  be a smooth function such that:

- (i)  $\|\nabla\varphi\| = 1$ ; and
- (ii) the level sets of  $\varphi$  are strictly convex with  $K$ -curvature greater than  $\kappa$ .

Then:

$$\Delta^K \varphi \geq -\|\text{Hess}(\varphi)\| \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

However, the maximum principal may be applied in a different way, allowing us to obtain existence results under different hypotheses. Let  $K$  be an admissible convex curvature function, and define  $\mu_\infty(K)$  by:

$$\mu_\infty(K) = \liminf_{A \rightarrow \infty, A \in \Gamma_1} DK_A(\text{Id}),$$

where  $\Gamma_1 := \{A \in \Gamma \text{ s.t. } K(A) = 1\}$ .

*Remark:* If  $K = \text{Det}$  is Gaussian curvature, then:

$$\mu_\infty(K) = +\infty,$$

and, if  $K = f_{n,k}$  is the curvature quotient, then:

$$\mu_\infty(K) = \left(\frac{n}{k}\right)^{1/(n-k)} > 1.$$

In general, by Proposition 2.3, (v),  $\mu_\infty(K) \geq 1$ .  $\square$

**Theorem 1.4**

Let  $M$  be a Hadamard manifold. Using the notation of Theorem 1.1, suppose, moreover, that there exists  $p \in M$  and  $R > 0$  such that:

- (i)  $\Sigma_l, \Sigma_u \subseteq B_R(p)$ ; and
- (ii)  $\kappa < \frac{1}{R} \mu_\infty(K)$ .

Then the conclusion of Theorem 1.1 continues to hold in this case.

**Theorem 1.5**

Let  $M$  be a Hadamard manifold. Using the notion of Theorem 1.2, suppose, moreover, that there exists  $p \in M$  and  $R > 0$  such that:

- (i)  $\Sigma_u \subseteq B_R(p)$ ; and
- (ii)  $\kappa < \frac{1}{R} \mu_\infty(K)$ .

Then the conclusion of Theorem 1.2 continues to hold in this case.



The following special case of Theorem 1.5 illustrates that these conditions are still fairly natural:

**Corollary 1.6**

Let  $K$  be an admissible convex curvature function of finite or determinant type; let  $\kappa : \mathbb{R}^{n+1} \rightarrow ]0, 1[$  be a smooth function; let  $S^n \subseteq \mathbb{R}^{n+1}$  be the unit sphere and let  $(\Omega, \partial\Omega) \subseteq S^n$  be a relatively compact open set with smooth boundary. Then, there exists a smooth, LSC, immersed hypersurface  $\Sigma$  in  $\mathbb{R}^{n+1}$  such that:

- (i)  $\Omega > \Sigma$ ; and
- (ii)  $K(\Sigma) = \kappa$ .

*1.5 The Plateau Problem in Affine Flat Manifolds.*

We now consider affine flat Hadamard manifolds where stronger existence results can be obtained. These are Hadamard manifolds which are everywhere locally affine equivalent to  $\mathbb{R}^n$ . Importantly, this class includes not only both Euclidean space and hyperbolic space as special cases, but also a large family which may be constructed by taking small deformations of hyperbolic space within the family of convex  $\mathbb{RP}$ -structures, as studied by Loftin in [16] and [17]. This is the natural context within which to apply the Perron method as developed by Guan and Spruck in [9] and Trudinger and Wang in [29], and we obtain:

**Theorem 1.7**

Let  $K$  be an admissible convex curvature function which is regular at infinity; let  $M$  be an  $(n+1)$ -dimensional affine flat Hadamard manifold; let  $\kappa \in C^\infty(M)$  be a smooth function; and let  $\Sigma_u = (\Sigma_u, \partial\Sigma_u)$  be a smooth, compact, LSC, immersed hypersurface such that:

- (i)  $K(\Sigma_u) > \kappa$ ; and
- (ii)  $\partial\Sigma_u$  only self intersects transversally.

Then there exists a  $C^{0,1}$ , compact, LSC, immersed hypersurface  $\Sigma := (\Sigma, \partial\Sigma)$  such that:

- (i)  $\Sigma$  is  $C^\infty$  away from the boundary;
- (ii)  $\Sigma_u > \Sigma$ ; and
- (iii)  $K(\Sigma) = \kappa$ .

*1.6 On Work of Guan, Spruck and Szapiel.*

Finally, we consider the asymptotic Plateau problem in  $\mathbb{H}^{n+1}$ , first studied by Rosenberg and Spruck for Gaussian curvature in [20], and then for more general convex curvature functions by Guan, Spruck and Szapiel in [10] (work which is then generalised further by Guan and Spruck in [11] to treat non-convex notions of curvature which fall outside the

scope of this paper). The machinery developed in this paper yields the following result which improves upon Theorem 1.2 of [8]:

**Theorem 1.8**

We identify  $\mathbb{H}^{n+1}$  with the upper half space in  $\mathbb{R}^{n+1}$ . Let  $\Gamma = \partial\Omega \subseteq \mathbb{R}^n$  where  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^n$ . Let  $K$  be an admissible convex curvature function such that  $K_\infty(1, \dots, 1) > 1$ . For all  $k \in ]0, 1[$ , there exists a complete, LSC,  $C^1$  hypersurface  $\Sigma$  in  $\mathbb{H}^{n+1}$  such that:

- (i)  $\Sigma$  is  $C^\infty$  away from the boundary;
- (ii)  $\partial\Sigma = \Gamma$ ; and
- (iii)  $K(\Sigma) = k$ .

Moreover, if  $u : \Omega \rightarrow [0, \infty[$  is the function of which  $\Sigma$  is the graph, then, for all  $p \in \partial\Omega$ :

$$\|Du(p)\|^2 = \frac{1}{k^2} - 1,$$

and, for all  $k \geq 1$ , there exists  $B_k > 0$  such that:

$$u^{k-1} \|D^k u\| < B_k.$$

*Remark:* This has been proven independantly using different techniques by Guan and Spruck in [12].  $\square$

*Remark:* We thus generalise completely to general convex curvature functions the result [20] of Rosenberg and Spruck, proven in that case for hypersurfaces of prescribed extrinsic curvature.  $\square$

*Remark:* This theorem is proven from Theorem 1.3 of [10] by taking limits, which is permitted by our Proposition 8.1. The condition that  $K_\infty(1, \dots, 1) > 1$  is imposed as one of the hypotheses used throughout [10]. In our setting, bearing in mind Theorem 1.7, we may replace it by the condition of being regular at infinity, although it is unclear which choice of conditions is preferable. In addition, Theorem 1.7 also allows us to extend Theorem 1.8 to the case where  $\Omega$  is non-simply connected and has a finite number of boundary components, all of which are smooth.  $\square$

*Remark:* Our Proposition 8.1, which is the key to this result, is a generalisation of Theorem 1.1 of [21] to more general ambient manifolds. However, in [21], Sheng, Urbas and Wang consider more general curvature functions than are studied here. We have nonetheless chosen to prove this result only for convex curvature functions since the context of the current paper doesn't justify the extra work. However, our experience suggests that there should be absolutely no obstacle to proving a generalisation of Sheng, Urbas and Wang's result in its entirety, and we believe that such a result may be used in conjunction with Theorem 1.3 of [11] to prove a corresponding generalisation of Theorem 1.8 to the more general curvature functions studied in that paper.  $\square$

## 1.7 Summary and Acknowledgements.

This paper is structured as follows:

- (i) in Sections 2 and 3 we introduce the concepts of convex curvature functions and convex cobordisms, and we study their basic properties;
- (ii) in Sections 4 to 8 we use barrier techniques to derive a-priori estimates for the derivatives of hypersurfaces of prescribed  $K$ -curvature, for general convex curvature functions,  $K$ . Sections 4 to 7, in particular, constitute the most innovative part of this paper where we show how straightforward barrier functions built from relatively simple geometric objects allow us to prove our results in the current generality;
- (iii) most of the theorems of this paper are proven in Section 9; and
- (iv) in Section 10, we show how the techniques developed in this paper allow us to prove Theorem 1.8.

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## 2 - Convex Curvature Functions.

We begin by proving the various properties of convex curvature functions which are required throughout the sequel.

### Proposition 2.1

If  $f$  is an admissible convex curvature function, then  $f_\infty$  is either everywhere infinite, or everywhere finite. Moreover, if  $f_\infty$  is everywhere finite, then:

- (i)  $f_\infty$  is homogeneous of order 1.
- (ii)  $f_\infty$  is concave; and
- (iii)  $f_\infty$  is (potentially non-strictly) elliptic;

**Proof:** The limit of an increasing family of concave functions over an open set is either everywhere infinite, or everywhere finite and concave. This proves the main assertion and Assertion (ii). Assertions (i) and (iii) then follow trivially.  $\square$

For  $1 \leq k < n$ , let  $f_{n,k}$  be the curvature quotient as defined in the introduction.

### Proposition 2.2

For  $1 \leq k < n$ ,  $f_{n,k}$  satisfies Axioms (i) to (vi) and (vii').

**Proof:** Denote  $f = f_{n,k}$ . (i) follows from the definition. (ii) and (iii) are trivial. Define  $\Phi : ]0, \infty[^n \rightarrow ]0, \infty[^n$  by:

$$\Phi(x_1, \dots, x_n) = (x^{-1}, \dots, x^{-n}).$$

Then:

$$f \circ \Phi = c_{n,k} \sigma_l^{-1/l},$$

where  $l = n - k$ . Trivially, if any component of  $x \in ]0, \infty[^n$  is infinite, then so is  $\sigma_l$  and so  $f$  extends to a continuous function over  $\bar{\Gamma}$  which vanishes over  $\partial\Gamma$ . This proves (iv). For  $x \in ]0, \infty[^n$  such that  $\sigma_l(x) = 1$ , if we denote  $y = \Phi(x)$ , then, for each  $i$ , by the chain rule:

$$(\partial_i f)(x) = \frac{1}{l} c_{n,k} y_i^2 \sigma_{l-1}(y_1, \dots, \hat{y}_i, \dots, y_n).$$

This is trivially positive, and (v) follows. (vi) is proven in [4]. Choose  $(y_1, \dots, y_n) \in ]0, \infty[^{n-1}$ . Trivially:

$$\lim_{t \rightarrow 0} \sigma_l(y_1, \dots, y_{n-1}, t) = \sigma_l(y_1, \dots, y_{n-1})$$

(vii')(a) now follows from (v). Finally:

$$\begin{aligned} \sigma_l(y_1, \dots, y_{n-1}, t) &= \sigma_l(y_1, \dots, y_{n-1}) + t \sigma_{l-1}(y_1, \dots, y_{n-1}) \\ \Rightarrow \sigma_l(y_1, \dots, y_{n-1}, t)^{-1/l} &= \sigma_l(y_1, \dots, y_{n-1})^{-1/l} - \frac{t}{l} \frac{\sigma_{l-1}(y_1, \dots, y_{n-1})}{\sigma_l(y_1, \dots, y_{n-1})} + O(t^2), \end{aligned}$$

for all sufficiently small  $t$ , and (vii')(b) follows. This completes the proof.  $\square$

Finally, we relate convex curvature functions to  $O(n)$  invariant functions over the space of symmetric matrices, and it is this perspective that will be most relevant to the sequel. Let  $\text{Symm}(n)$  be the space of real valued symmetric  $n$ -dimensional matrices and let  $\Gamma \subseteq \text{Symm}(n)$  be the open convex cone of positive definite matrices. Let  $f$  be an admissible convex curvature function. We define  $F \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$  by:

$$F(A) = f(\lambda_1, \dots, \lambda_n),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ .

### Proposition 2.3

Let  $f$  be an admissible curvature function. For all  $A \in \Gamma \subseteq \text{Symm}(n)$ , there exists a unique matrix  $B \in \text{Symm}(n)$  such that, for all  $M \in \text{Symm}(n)$ :

$$DF_A(M) = \text{Tr}(BM).$$

Moreover:

- (i)  $B$  is positive definite; and
- (ii)  $A$  and  $B$  are simultaneously diagonalisable.

In addition, if  $e_1, \dots, e_n$  is a system of shared eigenvectors for  $A$  and  $B$  and if  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_n$  are the corresponding eigenvalues of  $A$  and  $B$  respectively, then:

(iii) for all  $i \neq j$ :

$$\lambda_i \geq \lambda_j \Leftrightarrow \mu_i \leq \mu_j;$$

(iv)  $DF_A(A) = \sum_{i=1}^n \lambda_i \mu_i = F$ ;

(v)  $DF_A(\text{Id}) = \sum_{i=1}^n \mu_i \geq 1$ ; and

(vi) for all  $M \in \text{Symm}(n)$ :

$$-(D^2 f)_A(M, M) \geq \sum_{i=1}^n \frac{\mu_j - \mu_i}{\lambda_i - \lambda_j} M_{ij}^2 \geq 0.$$

In particular,  $F$  is concave.

**Proof:**  $DF_A : \text{Symm}(n) \rightarrow \text{Symm}(n)$  is linear. There therefore exists a unique matrix,  $B$ , such that, for all  $M \in \text{Symm}(n)$ :

$$DF_A(M) = \text{Tr}(BM).$$

By Axiom (i) of  $f$ ,  $F$  is  $O(n)$  invariant. Thus, for all  $A \in \Gamma$  and  $M \in O(n)$ :

$$F(M^t A M) = F(A).$$

Differentiating, for all antisymmetric  $M$ :

$$\begin{aligned} DF_A(MA - AM) &= 0 \\ \Rightarrow \text{Tr}([AB]M) &= 0. \end{aligned}$$

However, since  $A$  and  $B$  are both symmetric,  $[AB]$  is antisymmetric, and so, since  $M$  is arbitrary:

$$[AB] = 0.$$

(ii) now follows. Let  $e_1, \dots, e_n$  be a system of shared eigenvectors of  $A$  and  $B$  and let  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_n$  be the respective corresponding eigenvalues. By Axiom (i) of  $f$ , for all  $i$ :

$$\mu_i = DF_A(e_i \otimes e_i) > 0.$$

(i) now follows. (iii) follows by concavity of  $f$  and  $O(n)$ -invariance (Axioms (i) and (vi)) and (iv) follows by homogeneity (Axiom (ii)). Likewise, by concavity:

$$\begin{aligned} DF_A(\text{Id} - A) &\geq F(\text{Id}) - F(A) \\ \Rightarrow \sum_{i=1}^n \lambda_i - \sum_{i=1}^n \lambda_i \mu_i &\geq 1 - \kappa. \end{aligned}$$

Thus, by (iv):

$$\sum_{i=1}^n \lambda_i \geq 1.$$

(v) follows. Finally, (vi) follows by concavity as in Lemma 2.3 of [21]. This completes the proof.  $\square$

### 3 - Convex Cobordisms.

First order bounds are expressed in terms of compactness results for convex sets which we describe in this section. To this end, we study convex cobordisms in more detail. Trivially, when a convex cobordism exists between two LSC hypersurfaces, it is unique. In this section, we show that the property of existence of a convex cobordism is transitive. Compactness then follows.

Let  $M := M^{n+1}$  be an  $(n+1)$ -dimensional Hadamard manifold. For  $m \in \{1, 2\}$ , let  $\Sigma_m := (\Sigma_m, \partial\Sigma_m) = (i_m, (S_m, \partial S_m))$  be a compact, LSC, smooth, immersed hypersurface in  $M$ . Suppose that:

$$\Sigma_2 > \Sigma_1.$$

Let  $(I, N)$  be the cobordism from  $\Sigma_1$  to  $\Sigma_2$ . The geometry of  $N$  is encoded in the following:

#### Proposition 3.1

Choose  $p \in N \setminus S_1$ , let  $q \in S_1$  be a point minimising distance to  $p$  and let  $\gamma : [0, 1] \rightarrow N$  be a length minimising rectifiable curve such that  $\gamma(0) = q$  and  $\gamma(1) = p$ . Then:

- (i)  $\gamma([0, 1])$  lies in the interior of  $N$ ;
- (ii)  $\gamma([0, 1])$  is a smooth geodesic;
- (iii) if  $p \in S_2 \setminus \partial S_2$ , then  $\gamma$  is transverse to  $S_2$  at  $p$ ;
- (iv) if  $q \in S_1 \setminus \partial S_1$ , then  $\gamma$  makes a right angle with  $S_1$  at  $q$ ; and
- (v) if  $q \in \partial S_1$ , then  $\gamma$  is normal to  $\partial S_1$  at  $q$  and:

$$\langle (\partial_t \gamma)(0), N_{\partial S_1} \rangle \geq 0,$$

where  $N_{\partial S_1}$  is the normal vector to  $\partial S_1$  in  $S_1$  pointing outwards from  $S_1$ .

*Remark:* Importantly, the proof of this proposition does not require the existence of a foliation.  $\square$

**Proof:** Define  $t_0 \in [0, 1]$  by:

$$t_0 = \inf \{t \in [0, 1] \text{ s.t. } \gamma(t) \in S_2\}.$$

If  $q \in S_1 \setminus \partial S_1$ , then trivially  $t_0 > 0$ . If  $q \in \partial S_1$ , then likewise, by local strict convexity of  $S_2$ ,  $t_0 > 0$ .  $\gamma([0, t_0])$  lies inside  $S_2$ , and thus, by local strict convexity of  $S_2$ , it cannot be tangent to  $S_2$ . It follows that  $\gamma$  does not minimise length unless  $t_0 = 1, +\infty$  (the latter case occurring when  $\gamma$  never intersects  $S_2$ ) and this proves (i). (ii) is trivial. Likewise, taking  $t_0 = 1$ , we see that  $\gamma$  is transverse to  $S_2$  at  $\gamma(1) = p$ , and this proves (iii). (iv) and (v) are trivial, since  $\gamma$  is length minimising. This completes the proof.  $\square$

Let  $d : N \rightarrow \mathbb{R}$  be the distance in  $N$  to  $S_1$ .

#### Proposition 3.2

For all  $p \in N$ , there exist a unique point  $q \in S_1$  minimising distance to  $p$ .

**Proof:** Let  $q \in S_1$  minimise distance to  $p$ . Let  $\gamma : [0, 1] \rightarrow N$  be a length minimising curve from  $p$  to  $q$ . By Proposition 3.1, (i), (ii), (iv) and (v),  $\gamma$  is a geodesic in  $N$  which is normal to  $S_1$ . However, by hypotheses, there is only one such geodesic passing through  $p$ . This completes the proof.  $\square$

Let  $\pi : S_2 \rightarrow S_1$  be the projection onto the closest point. Since  $S_1$  is LSC and since  $N$  is non-positively curved,  $\pi$  is distance decreasing. We thus immediately obtain:

### Proposition 3.3

For each  $i$ , let  $\text{Diam}(\Sigma_i)$  and  $\text{Vol}(\Sigma_i)$  denote the diameter and volume of  $\Sigma_i$  respectively. Then:

$$\text{Diam}(\Sigma_2) \geq \text{Diam}(\Sigma_1), \quad \text{Vol}(\Sigma_2) \geq \text{Vol}(\Sigma_1).$$

Likewise, we obtain:

### Proposition 3.4

The function  $d$  is strictly convex.

**Proof:** Choose  $p \in N \setminus S_1$ . By Proposition 3.2, there exists a unique point  $q \in S_1$  minimising distance to  $p$ , and the result now follows trivially, since  $N$  is non-positively curved.  $\square$

We now prove transitivity. For  $m \in \{1, 2, 3\}$ , let  $\Sigma_m := (\Sigma_m, \partial\Sigma_m) = (i_m, (S_m, \partial S_m))$  be smooth, compact, LSC, immersed hypersurfaces in  $M$  and suppose that:

$$\Sigma_3 > \Sigma_2, \quad \Sigma_2 > \Sigma_1.$$

Moreover, let  $(I_{12}, N_{12})$  and  $(I_{23}, N_{23})$  be the convex cobordisms from  $\Sigma_1$  to  $\Sigma_2$  and from  $\Sigma_2$  to  $\Sigma_3$  respectively. We define  $N_{12} \cup N_{23}$  by joining  $N_{12}$  to  $N_{23}$  along  $\Sigma_2$ . We define  $I_{12} \cup I_{23} : N_{12} \cup N_{23} \rightarrow M$  by:

$$(I_{12} \cup I_{23})(x) = \begin{cases} I_{12}(x) & \text{if } x \in N_{12}; \text{ and} \\ I_{23}(x) & \text{if } x \in N_{23}. \end{cases}$$

### Proposition 3.5

There exists no non-trivial geodesic arc  $\gamma : [0, 1] \rightarrow N_{12} \cup N_{23}$  such that:

$$\gamma(0), \gamma(1) \in S_1.$$

**Proof:** Suppose the contrary. Let  $\gamma : [0, 1] \rightarrow N_{12} \cup N_{23}$  be a geodesic arc such that  $\gamma(0), \gamma(1) \in S_1$ . By Proposition 3.4,  $\gamma$  is not contained in  $N_{12}$ . There therefore exists  $t_0 \in ]0, 1[$  such that  $\gamma(t_0)$  lies in the interior of  $N_{23}$ , and there exists  $t_1 < t_0 < t_2$  such that  $\gamma([t_1, t_2])$  is contained in  $N_{23}$  and  $\gamma(t_1)$  and  $\gamma(t_2)$  both lie in  $\Sigma_2$ . However, this is also absurd by Proposition 3.4, and this completes the proof.  $\square$

### Proposition 3.6

$(I_{12} \cup I_{23}, N_{12} \cup N_{23})$  defines a convex cobordism from  $\Sigma_1$  to  $\Sigma_3$ .

**Proof:** Denote  $N = N_{12} \cup N_{23}$ . It suffices to show that for all  $p \in N$ , there exists a unique geodesic normal to  $S_1$  passing through  $p$ . Indeed, for  $p \in N \setminus S_3$ , let  $Q(p)$  be the set of points  $q \in S_1$  such that there exists a geodesic  $\gamma_q : [0, 1] \rightarrow N$  such that:

- (i)  $\gamma_q(0) = q$ ;
- (ii)  $\gamma_q$  is normal to  $S_1$  at  $q$ ; and
- (iii)  $\gamma_q(1) = p$ .

Since  $S_1$  is LSC,  $Q(p)$  is discrete. Since  $S_1$  is compact,  $Q(p)$  is therefore finite. Denote:

$$D(p) = |Q(p)|.$$

Choose  $q \in Q(p)$ . As is Proposition 3.1, (i),  $\gamma_q$  does not intersect  $S_2$  and by Proposition 3.5,  $\gamma_q$  only intersects  $S_1$  at  $q$ . It follows that, for all  $q \in Q(p)$ , and for all  $p'$  sufficiently close to  $p$ ,  $\gamma_q$  may be perturbed to another geodesic normal to  $S_1$  and terminating at  $p$ . It follows that  $D(p)$  is locally constant. However, by Proposition 3.5 again,  $D(p)$  is equal to 1 along  $S_1$ , and the result now follows by connectedness for  $p \notin S_2$ .

Finally, suppose  $p \in S_2$ . For  $q \in Q(p)$ , by Proposition 3.1, (iii),  $\gamma_q$  is transverse to  $S_2$  at  $p$ . We thus conclude as before that  $D$  is locally constant near  $p$  and thus  $D(p) = 1$  too. This completes the proof.  $\square$

Finally, let  $\Sigma := (\Sigma, \partial\Sigma) = (i, (S, \partial S))$  be another compact LSC smooth immersed hypersurface in  $M$  such that:

$$\Sigma_2 > \Sigma > \Sigma_1.$$

For  $\epsilon > 0$  and  $p \in S$ , we define  $S_{p,\epsilon}$  to be the connected component of  $i^{-1}(B_\epsilon(i(p)))$  containing  $p$  and we define  $\Sigma_{p,\epsilon}$  by:

$$\Sigma_{p,\epsilon} = (i, (S_{p,\epsilon}, \partial S_{p,\epsilon})).$$

By abuse of notation, we refer to  $\Sigma_{p,\epsilon}$  as the **connected component of  $\Sigma \cap B_\epsilon(p)$  containing  $p$** .

### Proposition 3.7

There exists  $\epsilon > 0$  which only depends on  $\Sigma_1$  and  $\Sigma_2$  such that for all  $p \in S$ ,  $\Sigma_{p,\epsilon}$  is embedded and lies on the boundary of a convex set.

**Proof:** Let  $(I_1, N_1)$  and  $(I_2, N_2)$  be the cobordisms from  $\Sigma_1$  to  $\Sigma$  and from  $\Sigma$  to  $\Sigma_2$  respectively. By Proposition 3.6,  $N_1 \cup N_2$  defines a cobordism from  $\Sigma_1$  to  $\Sigma_2$ . By definition,  $S$  is embedded in  $N$ . Trivially,  $I$  can be extended slightly to an immersion defined over an open (non complete) manifold containing  $N$ , and there exists  $\epsilon > 0$  such that, for all  $p \in N$ , the restriction of  $I$  to  $B_\epsilon(p)$  is an embedding. This is the desired value of  $\epsilon$ , and this completes the proof.  $\square$

Using the compactness of the family of convex sets, we immediately obtain the following compactness result:



### Proposition 3.8

Let  $(\Sigma_n)_{n \in \mathbb{N}}$  be a sequence of compact, LSC, smooth immersed hypersurfaces such that, for all  $n$ :

$$\Sigma_2 > \Sigma_n > \Sigma_1.$$

Then there exists a  $C^{0,1}$  LSC immersed hypersurface  $\Sigma_0$  such that:

- (i)  $(\Sigma_n)_{n \in \mathbb{N}}$  converges to  $\Sigma_0$  locally uniformly; and
- (ii)  $\Sigma_2 \geq \Sigma_0 \geq \Sigma_1$ ,

where  $\Sigma_2 \geq \Sigma_0$  means that  $\Sigma_2$  may be tangent to  $\Sigma_0$  at some points.

In addition, using the regularity of  $\partial\Omega$  we obtain a slightly stronger result for points near the boundary. Trivially, by compactness, there exists  $\epsilon > 0$  such that, for all  $p \in \Gamma := \partial\Sigma_1 = \partial\Sigma_2$  and for all  $q$  in  $B_\epsilon(p)$ , there exists a unique geodesic joining  $p$  to  $q$ . For such  $p$  and  $q$ , let  $\tau_{p,q}$  denote parallel transport from  $q$  to  $p$  along this geodesic. Observe that, since  $\Gamma$  is smooth, the normal to  $\Sigma_0$  at  $p$  is well defined and we denote it by  $N_0(p)$ . We obtain the following uniform modulus of continuity which is of critical importance throughout the sequel:

### Proposition 3.9

There exists a continuous function  $m : [0, 1] \times [0, \infty[ \rightarrow [0, \infty[$  such that  $m(0, 0) = 0$  and, for all  $\epsilon > 0$ , for all  $n \in \mathbb{N}$ , for all  $p \in \partial\Omega$ , and for all  $q \in B_\epsilon(p)$ , if  $N$  is a supporting normal to  $\Sigma_n$  at  $q$ , then:

$$D([N_{\Sigma_2}(p), N_0(p)], \tau_{p,q}(N)) < m(1/n, d(p, q)),$$

where  $D$  is the spherical distance in the unit sphere in  $T_q M$  and  $[N_{\Sigma_2}(p), N_0(p)]$  is the geodesic segment joining  $N_{\Sigma_2}(p)$  to  $N_0(p)$  in the unit sphere in  $T_q M$ .

*Remark:* Importantly,  $N_0$  is not necessarily continuous along  $\Gamma$ . Indeed, consider the following convex graph: let  $B_1(0)$  be the ball of radius 1 about the origin in  $\mathbb{R}^n$ . Let  $v$  be a unit vector in  $\mathbb{R}^n$  and, for  $\alpha \in ]0, 1[$ , define  $f : B_1(0) \rightarrow \mathbb{R}$  by:

$$f(x) = \text{Min}(\alpha \langle x, v \rangle + \alpha, 1 - \|x\|).$$

The function  $f$  is concave and vanishes along  $\partial B_1(0)$ . For  $x \in \partial B_1(0)$ , if  $N$  is the supporting normal to the graph of  $f$  at  $x$ , then  $N(x)$  is well defined for all  $x$ , and:

$$N(x) = \begin{cases} \frac{1}{\sqrt{2}}(x, 1) & \text{if } x \neq v, \\ (1 + \alpha^2)^{-1/2}(\alpha v, 1) & \text{if } x = v. \end{cases}$$

This is trivially not continuous.

**Proof:** The set of supporting normals to an LSC hypersurface at any point is a continuous function with respect to the Hausdorff topology of the convergence of sets. Likewise, for a sequence of convex sets converging towards a limit, the supporting normal sets subconverge to a subset of the supporting normal set of the limit. Observe that, since  $\Sigma_2$  is smooth, we may extend it smoothly beyond its boundary by adjoining an LSC collar region,  $\Sigma_{2,c}$ . Then  $\Sigma_0 \cup \Sigma_{2,c}$  is a  $C^{0,1}$  immersed hypersurface, and, for all  $p \in \Gamma$ , its set of supporting normals at  $p$  is  $[N_{\Sigma_2}, N_0(p)]$ . The result now follows by compactness, and this completes the proof.  $\square$

## 4 - First Order Lower Estimates Along The Boundary.

Let  $M := M^{n+1}$  be an  $(n+1)$ -dimensional manifold of non-positive curvature. Let  $K$  be an admissible convex curvature function. In this and the following section, we will suppose that  $K$  satisfies Axiom (vii') (and so  $K_\infty$  is everywhere finite), and we obtain lower estimates for the normals of locally convex hypersurfaces of prescribed  $K$ -curvature. Explicitly, let  $\kappa \in C^\infty(M)$  be a smooth, strictly positive function, let  $\Sigma_l := (\Sigma_l, \partial\Sigma_l)$  and  $\Sigma_u := (\Sigma_u, \partial\Sigma_u)$  be smooth, LSC hypersurfaces with smooth boundary such that:

- (i)  $\Sigma_u > \Sigma_l$ ; and
- (ii)  $K(\Sigma_l) < \kappa < K(\Sigma_u)$ .

In the sequel, we denote by  $\mathcal{B}$  the family of all quantities which depend continuously only upon the data, being, in this case  $M$ ,  $K$ ,  $\kappa$ ,  $\Sigma_l$  and  $\Sigma_u$ . For any supplementary data,  $X$ , we denote by  $\mathcal{B}(X)$  the family of quantities which also depend on  $X$ .

For  $\mathbf{N}$  a normal vector to  $\Gamma$ , let  $A_\Gamma(\mathbf{N})$  be the second fundamental form of  $\Gamma$  in the direction of  $\mathbf{N}$ . In other words, if  $X$  and  $Y$  are vector fields tangent to  $\Gamma$ :

$$A_\Gamma(X, Y) = -\langle \nabla_X Y, \mathbf{N} \rangle.$$

We say that  $\Gamma$  is strictly convex with respect to  $V$  if and only if  $A_\Gamma(V)$  is positive definite. We now consider how the normal vectors are configured (see Figure 1).

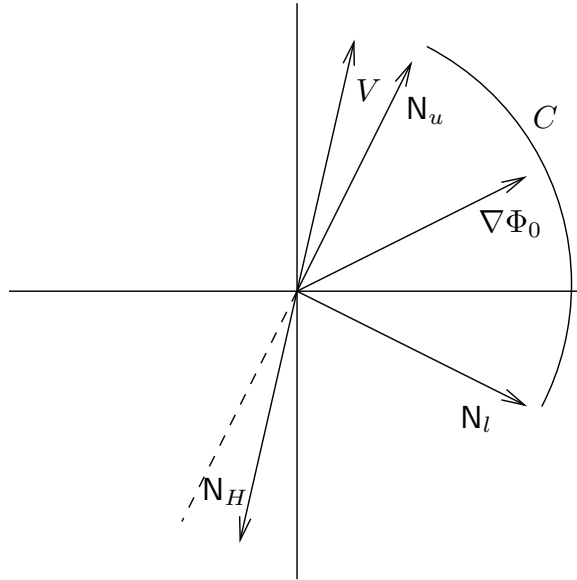


Figure 1

Let  $UM$  be the bundle of unit tangent vectors over  $M$ . Denote  $\mathbf{N}_u := \mathbf{N}_{\Sigma_u}$  and  $\mathbf{N}_l = \mathbf{N}_{\Sigma_l}$ . Consider  $p \in \Gamma := \partial\Sigma_u = \partial\Sigma_l$ . Since  $\Sigma_u$  and  $\Sigma_l$  are both convex,  $\mathbf{N}_u(p) \neq -\mathbf{N}_l(p)$ . We

thus define  $C(p)$  to be the unique shortest geodesic in  $U_p M$  joining  $N_u(p)$  to  $N_l(p)$ . By Proposition 3.9, if  $\Sigma$  is a  $C^{0,1}$  LSC hypersurface such that  $\Sigma_l < \Sigma < \Sigma_u$ , then, for any point,  $q$ , sufficiently close to  $p$ , every supporting normal to  $\Sigma$  at  $q$  lies close to  $C(p)$ .

Choose  $V \in U_p M$  such that:

- (i)  $V$  is coplanar with  $N_l$  and  $N_u$ ;
- (ii)  $V \notin C$ ; and
- (iii)  $\Gamma$  is strictly convex with respect to  $V$ .

For  $V$  sufficiently close to  $N_u(p)$ , we may define the embedded hypersurface  $H$  such that:

- (i)  $H$  passes through  $p$  and its outward pointing normal at  $p$  is  $N_H = -V$ ;
- (ii)  $H$  is strictly concave; and
- (iii)  $\Gamma$ ,  $\Sigma_u$  and  $\Sigma_l$  lie locally above  $H$ .

This configuration yields a barrier function which allows us to prove:

#### Proposition 4.1

There exists  $\delta > 0$  in  $\mathcal{B}$  such that if  $\Sigma := (\Sigma, \partial\Sigma)$  is a smooth, LSC immersed hypersurface in  $M$  such that:

- (i)  $\Sigma_u > \Sigma > \Sigma_l$ ; and
- (ii)  $K(\Sigma) = \kappa$ ;

and if  $N_\Sigma$  is the outward pointing unit normal over  $\Sigma$ , then, over  $\Gamma := \partial\Sigma$ :

$$K_\infty(A_\Gamma(N_\Sigma)) \geq \kappa + \delta.$$

*Remark:* For  $N$  a normal vector to  $\Gamma$ , define  $\hat{K}(N)$  by:

$$\hat{K}(N) = K_\infty(A_\Gamma(N)).$$

Trivially,  $\hat{K}$  is a continuous function of the set of normal vectors of  $\Gamma$  with respect to which  $\Gamma$  is strictly convex. Proposition 4.1 tells us that the normal to  $\Sigma$  lies uniformly in the complement of  $\hat{K}^{-1}([0, \kappa(p)])$  for all  $p \in \Gamma$ .  $\square$

Proposition 4.1 is proven in the following section. In this section we construct the barrier. Choose  $p \in \Gamma$ . Let  $N_p$  be a normal vector to  $\Gamma$  at  $p$  lying between the outward normals of  $\Sigma_l$  and  $\Sigma_u$ . Let  $\lambda_1, \dots, \lambda_{n-1}$  be the eigenvalues of  $A_\Gamma(N_p)$ , and suppose that:

$$K_\infty(\lambda_1, \dots, \lambda_{n-1}) = \kappa(p).$$

Let  $\Phi_0$  be a smooth function defined in a neighbourhood of  $p$  such that:

- (i)  $\nabla\Phi_0 = N_p$ ; and
- (ii)  $\Phi_0$  vanishes over  $\Gamma$ .

The barrier will be constructed by perturbing  $\Phi_0$ . We define  $d_H$  by:

$$d_H(q) = d(q, H).$$

For any two functions  $f$  and  $g$  with non-colinear derivatives at  $p$ , we define the  $(n-2)$ -dimensional distribution  $E(f, g)$  near  $p$  by:

$$E(f, g) = \langle \nabla f, \nabla g \rangle^\perp,$$

where  $\langle X, Y \rangle$  here denotes the space spanned by the vectors  $X$  and  $Y$ . Let  $e_1, \dots, e_{n-1}$  be an orthonormal basis for  $T_p \Gamma$  with respect to which  $A_\Gamma(\mathbf{N}_p)$  is diagonal. Bearing in mind that  $\nabla d_H$  and  $\nabla \Phi_0$  are non-colinear at  $p$ , we extend this to a local frame in  $TM$  such that, at  $p$ , for all  $X$  and all  $i$ :

$$\begin{aligned} \langle \nabla_X e_i, \nabla d_H \rangle &= -\text{Hess}(d_H)(e_i, X), \\ \langle \nabla_X e_i, \nabla \Phi_0 \rangle &= -\text{Hess}(\Phi_0)(e_i, X). \end{aligned}$$

Define the distribution  $E$  near  $p$  to be the span of  $e_1, \dots, e_{n-1}$ .

### Proposition 4.2

If  $D$  represents the Grassmannian distance between two  $(n-2)$ -dimensional spaces, then:

$$D(E, E(\Phi_0, d_H)) = O(d_p^2).$$

**Proof:** Indeed, by definition, for all  $X$  and for all  $i$ , at  $p$ :

$$X \langle e_i, \nabla d_H \rangle = \langle \nabla_X e_i, \nabla d_H \rangle + \langle e_i, \nabla_X \nabla d_H \rangle = 0.$$

Likewise:

$$X \langle e_i, \nabla \Phi_0 \rangle = 0.$$

The result follows.  $\square$

For a smooth function  $f$ , we define  $K_{\infty, E}(f)$  by:

$$K_{\infty, E}(f) = \frac{1}{\|\nabla f\|} K_\infty(\text{Hess}(f)|_E),$$

where  $\text{Hess}(f)|_E$  is the restriction to  $E$  of the Hessian of  $f$ .

### Proposition 4.3

If  $V$  is sufficiently close to  $\mathbf{N}_u$ , then, for all  $f$  defined near  $p$  such that:

- (i)  $f(p) = 0$ ;
  - (ii)  $\nabla f(p) = 0$ ;
  - (iii) the Hessian of the restriction of  $f - d_H$  to  $\Gamma$  vanishes at  $p$ ; and
  - (iv) the restriction of  $\text{Hess}(f)$  to  $H$  is positive definite,
- there exists a function  $x$  such that  $x(p), \text{Hess}(x)(p) = 0$  and:

$$K_{\infty, E}(\Phi_0 + x(d_H - f)) \leq \kappa + O(d_p^2).$$

**Proof:** Denote  $\Phi_1 = \Phi_0 + x(d_H - f)$ . By definition, the restriction of  $\|\nabla\Phi_0\|^{-1}\text{Hess}(\Phi_0)$  to  $E$  coincides with  $A_\Gamma(\mathbf{N}_p)$  and thus, by definition of  $\mathbf{N}_p$ , at  $p$ :

$$K_{\infty,E}(\Phi_0) = \kappa(p).$$

The gradient of  $x(d_H - f)$  vanishes at  $p$ , the Hessian of  $xf$  vanishes at  $p$ , and the Hessian of the second order term  $xd_H$  vanishes on  $(\nabla d_H)^\perp$  at  $p$  and thus so does its restriction to  $E$ . It thus follows that  $x(d_H - f)$  does not affect  $K_{\infty,E}$  at  $p$ , and so, for all  $x$ , at  $p$ :

$$K_{\infty,E}(\Phi_1) = \kappa(p).$$

Let  $\lambda_1, \dots, \lambda_{n-1}$  be the eigenvalues of the restriction of  $\text{Hess}(\Phi_0)$  to  $E$  at  $p$ . Let  $(\mu_1, \dots, \mu_{n-1})$  be a supporting tangent to  $K_\infty$  at  $(\lambda_1, \dots, \lambda_{n-1})$ . By strict ellipticity of  $K_\infty$  (Axiom (vii')(a)), we may assume that, for all  $k$ :

$$\mu_k > 0.$$

Suppose first that all the  $\lambda_i$  are distinct. Define  $\tilde{K}_\infty$  such that, for all  $\lambda'_1, \dots, \lambda'_{n-1}$ :

$$\tilde{K}_\infty(\lambda'_1, \dots, \lambda'_{n-1}) := K_\infty(\lambda_1, \dots, \lambda_{n-1}) + \sum_{i=1}^n \mu_i(\lambda'_i - \lambda_i).$$

By concavity of  $K_\infty$  (Proposition 2.1, (ii)), for all  $\lambda'_1, \dots, \lambda'_{n-1}$ :

$$K_\infty(\lambda'_1, \dots, \lambda'_{n-1}) \leq \tilde{K}_\infty(\lambda'_1, \dots, \lambda'_{n-1}).$$

Using  $\tilde{K}_\infty$  instead of  $K_\infty$ , we define  $\tilde{K}_{\infty,E}$  in the same way as  $K_\infty(E)$ . Denote  $P = x(d_H - f)$ . At  $p$ :

$$\text{Hess}(P) = \nabla x \otimes \nabla d_H + \nabla d_H \otimes \nabla x.$$

At  $p$ , for all  $i$ , by definition,  $\langle e_i, \nabla d_H \rangle = 0$ . Thus, recalling the formula for  $\nabla e_i$ , for all  $X$ :

$$\begin{aligned} X\text{Hess}(P)(e_i, e_j) &= (\nabla_X \text{Hess}(P))(e_i, e_j) + \text{Hess}(P)(\nabla_X e_i, e_j) + \text{Hess}(P)(e_i, \nabla_X e_j) \\ &= (\nabla_X \text{Hess}(P))(e_i, e_j) - \text{Hess}(d_H)(X, e_i)x_{;j} - \text{Hess}(d_H)(X, e_j)x_{;i}. \end{aligned}$$

We extend  $e_1, \dots, e_{n-1}$  to a basis  $e_0, \dots, e_n$  for  $T_p M$ . Observe, in particular, that the plane spanned by  $e_0$  and  $e_n$  coincides with the plane spanned by  $\mathbf{N}_p$  and  $\nabla d_H$ . Since all the  $\lambda_i$  are distinct, they are smooth in a neighbourhood of  $p$  and, thus, with respect to this basis, for all  $i, k$ , bearing in mind the terms that vanish at  $p$ , we obtain:

$$\begin{aligned} \partial_k \lambda_i &= \partial_k \text{Hess}(\Phi_0)(e_i, e_i) + \partial_k \text{Hess}(P)(e_i, e_i) \\ &= \partial_k \text{Hess}(\Phi_0)(e_i, e_i) - 2x_{;i}f_{;ik} - x_{;k}f_{;ii} + x_{;k}d_{H;ii}. \end{aligned}$$

However, since the Hessian of the restriction of  $f - d_H$  to  $\Gamma$  vanishes at  $P$ , and since  $\nabla(f - d_H)(p) = \nabla - d_H(p) = V$  is a unit normal to  $\Gamma$  at  $p$ , we obtain, for all  $i, j$  at  $p$ :

$$f_{;ii} - d_{H;ii} = A_\Gamma(-\nabla d_H)_{ii} = A_\Gamma(V)_{ii}.$$

Thus, for all  $i, k$ , at  $p$ :

$$\partial_k \lambda_i = \partial_k \text{Hess}(\Phi_0)(e_i, e_i) - 2x_{;i} f_{;ik} - x_{;k} A_\Gamma(V)_{ii}.$$

Thus, at  $p$ , for all  $k$ , bearing in mind that  $\tilde{K}_{\infty, E}(\Phi_1) = \kappa$  at  $p_0$ :

$$\partial_k \tilde{K}_{\infty, E}(\Phi_1) = \partial_k \tilde{K}_{\infty, E}(\Phi_0) + (M \nabla x)_k,$$

where, for any vector  $U$ :

$$\begin{aligned} (MU)_k &= -\kappa \langle \nabla \Phi_0, \nabla d_H \rangle U_k - \kappa \langle \nabla \Phi_0, U \rangle d_{H;k} \\ &\quad - 2 \sum_{i=1}^{n-1} \mu_i U_i f_{;ik} - \sum_{i=1}^{n-1} \mu_i U_k A_\Gamma(V)_{ii}. \end{aligned}$$

We claim that, after modifying  $V$  slightly if necessary,  $M$  is invertible. Indeed, suppose that  $MU = 0$  for some non-trivial  $U$ . Taking the inner product with  $(0, \mu_1 U_1, \dots, \mu_{n-1} U_{n-1}, 0)$  yields:

$$2 \sum_{i,k=1}^{n-1} (\mu_i U_i)(\mu_k U_k) f_{;ik} + (\kappa \langle \nabla \Phi_0, \nabla d_H \rangle + \sum_{i=1}^{n-1} \mu_i A_\Gamma(V)_{ii}) \sum_{k=1}^{n-1} \mu_k U_k^2 = 0$$

However, choosing  $V$  sufficiently close to  $N_u$ , and bearing in mind concavity and Proposition 2.3, (iv) applied to  $k_\infty$ :

$$\begin{aligned} \kappa &< K_\infty(A_\Gamma(V)) \\ &\leq \kappa + \sum_{i=1}^{n-1} \mu_i (A_\Gamma(V)_{ii} - \lambda_i) \\ &= \sum_{i=1}^{n-1} \mu_i A_\Gamma(V)_{ii}. \end{aligned}$$

Thus:

$$\kappa \langle \nabla \Phi_0, \nabla d_H \rangle + \sum_{i=1}^{n-1} \mu_i A_\Gamma(V)_{ii} > 0,$$

and so, since  $\mu_k > 0$  for all  $k$ :

$$\begin{aligned} \sum_{i,k=1}^{n-1} (\mu_i U_i)(\mu_k U_k) f_{;ik} &= 0 \\ \Rightarrow U_k &= 0 \text{ for all } 1 \leq k \leq n-1, \end{aligned}$$

since  $f_{;ij}$  is positive definite. Now observe that  $M$  preserves  $\langle \nabla d_H, \nabla \Phi_0 \rangle = \langle e_0, e_n \rangle$ . With respect to the basis  $(\nabla d_H, \nabla \Phi_0)$ , and recalling that  $\nabla d_H = -V$ , the matrix of the restriction of  $M$  is given by:

$$M|_{\langle \nabla d_H, \nabla \Phi_0 \rangle} = - \begin{pmatrix} \lambda(V) - 2\kappa \langle \nabla \Phi_0, V \rangle & \kappa \\ \lambda(V) - \kappa \langle \nabla \Phi_0, V \rangle & \end{pmatrix},$$

where, for all  $U$ :

$$\lambda(U) = \sum_{i=1}^n \mu_i A_\Gamma(U)_{ii}.$$

## The Plateau Problem for General Curvature Functions

Let  $U$  be a unit vector in the plane spanned by  $\nabla\Phi_0$  and  $V = -\nabla d_H$ , and let  $\theta$  be the angle between  $\nabla\Phi_0$  and  $U$ . There exists  $a, b \in \mathbb{R}$  such that:

$$\lambda(U) - 2\kappa\langle\nabla\Phi_0, U\rangle = (a - 2\kappa)\cos(\theta) + b\sin(\theta).$$

When  $\theta = 0$ , bearing in mind Proposition 2.3, (iv):

$$\begin{aligned} \Rightarrow \quad & \sum_{i=1}^{n-1} \mu_i A_\Gamma(U)_{ii} = \sum_{i=1}^{n-1} \mu_i \lambda_i \\ \Rightarrow \quad & \lambda = \kappa. \end{aligned}$$

It follows that  $a - 2\kappa = \kappa \neq 0$ . Thus, by varying  $V$  slightly if necessary, we may assume that:

$$\lambda(V) - 2\kappa\langle\nabla\Phi_0, V\rangle \neq 0.$$

Finally, we have already shown that  $\lambda - \kappa\langle\nabla\Phi_0, V\rangle > 0$ , and so the restriction of  $M$  to  $\langle\nabla d_H, \nabla\Phi_0\rangle$  is therefore invertible. Thus  $U_0 = U_n = 0$ , and  $M$  is therefore invertible as asserted.

Since  $M$  is invertible, there exists  $x$  such that, at  $p$ , for all  $k$ :

$$(M\nabla x)_{;k} = -\partial_k \tilde{K}_{\infty, E}(\Phi_0) + \kappa_{;k}.$$

Consequently:

$$\begin{aligned} \tilde{K}_{\infty, E}(\Phi) &= \kappa + O(d_p^2) \\ \Rightarrow \quad K_{\infty, E}(\Phi) &\leq \kappa + O(d_p^2). \end{aligned}$$

Finally, if  $\lambda_i = \lambda_j$  for some  $i \neq j$ , then, by convexity, we may choose  $\mu$  such that  $\mu_i = \mu_j$ . We then proceed as before, and this completes the proof.  $\square$

For  $M > 0$ , we define  $\Phi$  by:

$$\Phi = \Phi_0 + x(d_H - f) + Md_H^2.$$

### Proposition 4.4

If  $D$  represents the Grassmannian distance between two  $(n-2)$ -dimensional subspaces then:

$$D(E(\Phi_0, d_H), E(\Phi, d_H)) = O(d_p^2) + O(d_H).$$

**Proof:** Since  $xf$  is of order 3, near  $p$ :

$$\nabla\Phi = \nabla\Phi_0 + (x + 2Md_H)\nabla d_H + O(d_p^2) + O(d_H).$$

Thus:

$$\langle\nabla\Phi, \nabla d_H\rangle = \langle\nabla\Phi_0 + O(d_p^2) + O(d_H), \nabla d_H\rangle,$$

where  $\langle\cdot, \cdot\rangle$  here represents the subspace generated by two vectors. The result follows.  $\square$

### Corollary 4.5

If  $D$  represents the Grassmannian distance between two  $(n-2)$ -dimensional subspaces then:

$$D(E, E(\Phi, d_H)) = O(d_p^2) + O(d_H).$$

**Proof:** This follows by the triangle inequality and Proposition 4.2.  $\square$

**Proposition 4.6**

Let  $\epsilon > 0$ . If  $M\epsilon^2 < 1$ ,  $d_H < \epsilon^2$  and  $d_p < \epsilon$ , then either:

- (i) the restriction of  $\text{Hess}(\Phi)$  to  $E(\nabla\Phi, \nabla d_H)$  is not positive definite; or
- (ii)

$$K_{\infty, E(\nabla\Phi, \nabla d_H)}(\Phi) \leq \kappa + O(\epsilon^2).$$

**Proof:** Define  $\Phi_1$  by:

$$\Phi_1 = \Phi_0 + x(d_H - f).$$

By Proposition 4.3:

$$K_{\infty, E}(\Phi_1) \leq \kappa + O(\epsilon^2).$$

Since  $\text{Hess}(\Phi_1) = O(1)$ , by Corollary 4.5:

$$K_{\infty, E(\nabla\Phi, \nabla d_H)}(\Phi_1) \leq \kappa + O(\epsilon^2).$$

Differentiating  $Md_H^2$  yields:

$$\text{Hess}(Md_H^2) = 2M\nabla d_H \otimes \nabla d_H + 2Md_H \text{Hess}(d_H).$$

The first term vanishes along  $(\nabla d_H)^\perp$ . The second term is negative definite. Consequently, either the restriction of  $\text{Hess}(\Phi)$  to  $E(\nabla\Phi, \nabla d_H)$  is not positive definite, or it is, and this term does not affect the inequality. The result now follows.  $\square$

## 5 - Constructing the Barrier.

Let  $M$ ,  $H$ ,  $\Sigma_l$  and  $\Sigma_u$  be as in the preceeding section. Let  $\Sigma$  be a  $C^{0,1}$  LSC immersed hypersurface such that  $\Sigma_l \leq \Sigma \leq \Sigma_u$  and whose interior is a viscosity solution of  $K(\Sigma) = \kappa$ . Since  $\Gamma := \partial\Sigma_l = \partial\Sigma$  is smooth,  $\Sigma$  has a well defined outward pointing unit normal,  $N_p$ , at  $p$ .

Proposition 4.1 follows from the following result:

**Proposition 5.1**

if  $\lambda_1, \dots, \lambda_{n-1}$  are the principal curvatures of  $\Gamma$  with respect to the normal,  $N_p$ , at  $p$ , then:

$$K_\infty(\lambda_1, \dots, \lambda_{n-1}) > \kappa(p).$$



**Proof:** We assume the contrary. Thus  $K_\infty(\lambda_1, \dots, \lambda_{n-1}) = \kappa(p)$ . Define  $d_p$ ,  $d_H$  and  $\Phi_0$  as in the previous section. For  $\epsilon > 0$  small, define  $U_\epsilon$  by:

$$U_\epsilon = \{q \in M \text{ s.t. } d_p(q) < \epsilon, d_H(q) < \epsilon^2\}.$$

Let  $X$  be a field of unit vectors defined near  $p$  such that  $X(p) = \mathbf{N}_u(p)$ . For  $q \in M$ , let  $U_q M$  be the unit sphere in  $T_q M$ . Let  $D_q$  be the distance in  $U_q M$  and let  $C_q$  now be the shortest geodesic in  $U_q M$  joining  $X(q)$  to  $(\|\nabla \Phi_0\|^{-1} \nabla \Phi_0)(q)$ . Near  $p$ ,  $X(p) = \mathbf{N}_u(p)$ ,  $\nabla \Phi_0$  and  $\nabla d_H$  are configured as in Figure 1 (observe, however, that  $C$  now only extends from  $\mathbf{N}_u$  to  $\nabla \Phi_0$ ).

However, by definition,  $\mathbf{N}_p$  equals  $\nabla \Phi_0$ . Thus, by Proposition 3.9, there exists a continuous function  $\delta : [0, \infty[ \rightarrow [0, \infty[$  such that  $\delta(0) = 0$  and, for all  $q \in \Sigma$ , if  $\mathbf{N}_q$  is a supporting normal to  $\Sigma$  at  $q$ , then:

$$D_q(\mathbf{N}_q, C_q) \leq \delta(d_p(q)).$$

Thus, if, for all  $q \in \Sigma$ ,  $\pi_q$  is a projection onto a supporting hyperplane of  $\Sigma$  at  $q$ , then:

(i) there exists  $c > 0$  such that, for all  $q$  sufficiently close to  $p$ :

$$\|\pi_q(\nabla d_H)\| \geq c.$$

(ii) for all  $q$  sufficiently close to  $p$ :

$$\langle \pi_q(\nabla \Phi_0), \pi_q(\nabla d_H) \rangle \geq -\delta(d_p(q)).$$

Now consider  $q \in \Sigma \cap \partial U$ . Let  $\gamma : I \rightarrow \Sigma$  be an integral curve of  $\pi_q(\nabla d_H)$  such that  $\gamma(0) \in \partial \Sigma$  and  $\gamma(1) = q$  (which is defined by approximating  $\Sigma$  by smooth hypersurfaces). Bearing in mind that  $\Phi_0$  vanishes along  $\partial \Sigma$ :

$$\begin{aligned} d_H(q) &\leq \epsilon^2 \\ \Rightarrow \text{Length}(\gamma) &\leq \epsilon^2 c^{-1} \\ \Rightarrow (\Phi \circ \gamma)(1) &\geq -\delta(\epsilon) \epsilon^2 c^{-1}. \end{aligned}$$

Thus:

$$[\Phi_0(q) + x(d_H - f)](q) \geq -\delta(\epsilon) O(\epsilon^2),$$

for all appropriate functions  $f$  and  $x$ . Since  $\Gamma$  is strictly convex and lies strictly inside  $H$ , we may choose  $f$  such that:

- (i)  $f(p), \nabla f(p) = 0$  and the restriction of  $\text{Hess}(f)(p)$  to  $H$  is positive definite; and
- (ii)  $d_H - f = O(d_p^3)$  along  $\Gamma$ .

We define  $\Phi$  as in the previous section. Along  $\partial \Sigma \cap U_\epsilon = \Gamma \cap U_\epsilon$ :

$$\Phi(q) \geq M d_H^2 - O(\epsilon^4).$$

This is positive for sufficiently large  $M$ . Likewise, along  $\partial U_\epsilon \cap \Sigma$ :

$$\Phi(q) \geq M d_H^2 - \delta(\epsilon) O(\epsilon^2).$$

There thus exists  $K_1 > 0$  independent of  $\epsilon$  such that, if  $M = K_1\delta(\epsilon)\epsilon^{-2}$ , then  $\Phi \geq 0$  along  $\partial U_\epsilon \cap \Sigma$ .

Let  $A$  be the restriction of  $\|\nabla\Phi\|^{-1}\text{Hess}(\Phi)$  to  $\nabla\Phi^\perp$  and suppose that  $A$  is positive definite (and so the level set of  $\Phi$  passing through this point is convex). Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $A$ . Let  $\lambda'_1, \dots, \lambda'_{n-1}$  be the eigenvalues of the restriction of  $A$  to  $\nabla\Phi^\perp \cap \nabla d_H^\perp$ . By the Minimax Principal (see, for example, Lemma 10.2 of [25]), for  $1 \leq i \leq (n-1)$ :

$$\lambda_i \leq \lambda'_i.$$

Thus, by Proposition 4.6 and ellipticity of  $K_\infty$  (Axiom (vii')(a)), there exists  $K_2 > 0$ , also independent of  $\epsilon$  such that:

$$K_\infty(\lambda_1, \dots, \lambda_{n-1}) \leq K_\infty(\lambda'_1, \dots, \lambda'_{n-1}) \leq \kappa + K_2\epsilon^2.$$

However,  $\lambda_n \leq O(M)$ . Thus, by Axiom (vii')(b) of  $K$ , there exists  $K_3 > 0$ , independent of  $\epsilon$ , such that:

$$\begin{aligned} K(A) &\leq \kappa + (K_2\epsilon^2 - K_3M^{-1}) \\ &= \kappa + \epsilon^2(K_2 - K_1K_3\delta(\epsilon)^{-1}). \end{aligned}$$

Since  $\delta(\epsilon)$  tends to 0 as  $\epsilon$  tends to 0, there exists  $\eta > 0$  such that, for  $\epsilon$  sufficiently small, throughout  $U_\epsilon$ :

$$K(A) \leq \kappa - \eta < \kappa.$$

It follows that if  $\Sigma_t = \Phi^{-1}(\{t\})$  for all  $t$ , then, at every point of  $\Sigma_t \cap U_\epsilon$  where this hypersurface is convex:

$$K(\Sigma_t) \leq \kappa - \eta < \kappa.$$

At  $p$ ,  $\nabla^\Sigma\Phi = 0$ . Thus, reducing  $\epsilon$  further if necessary, we may deform  $\Phi$  slightly to  $\Phi'$  such that  $\Phi'$  is non-negative along  $\partial\Sigma \cap U_\epsilon$ ,  $\Phi$  is strictly negative over a non trivial subset of  $\Sigma \cap U_\epsilon$ , and, if  $\Sigma'_t = (\Phi')^{-1}(\{t\})$  for all  $t$ , then, at every point of  $\Sigma'_t \cap U_\epsilon$  where this hypersurface is convex:

$$K(\Sigma'_t \cap U_\epsilon) \leq \kappa - \eta/2 < \kappa.$$

Let  $p \in \Sigma$  be the point where  $\Phi$  is minimised. Let  $t_0 = \Phi(p)$ . Since  $p$  lies in the interior of  $\Sigma$ ,  $\Sigma$  is a viscosity solution of  $K(\Sigma) = \varphi$  at this point. However,  $\Sigma'_{t_0}$  is an interior tangent to  $\Sigma$  at  $p$ , and so  $\Sigma'_{t_0}$  is convex near  $p$ , and:

$$K(\Sigma'_{t_0} \cap U_\epsilon) < \kappa,$$

which is absurd by the Geometric Maximum Principal, and the result follows.  $\square$

**Proof of Proposition 4.1:** Suppose the contrary. Let  $(\Sigma_m)_{m \in \mathbb{N}}$  be a sequence of strictly convex immersed hypersurfaces in  $M$  which are graphs over  $\Sigma_0$  satisfying conditions (i) and (ii) of Proposition 4.1. Suppose, moreover, that, for all  $m$ , we may choose  $p_m \in \Gamma$  such that if  $\mathbf{N}_m$  is the outward pointing unit normal to  $\Sigma_m$  at  $p_m$ , and if  $\lambda_{1,m}, \dots, \lambda_{n,m}$  are the principal curvatures of  $\Gamma$  with respect to this normal at  $p_m$ , then:

$$(K_\infty(\lambda_{1,m}, \dots, \lambda_{n,m}))_{m \in \mathbb{N}} \rightarrow \kappa(p).$$

By Proposition 3.8, we may suppose that  $(\Sigma_m)_{m \in \mathbb{N}}$  converges to a  $C^{0,1}$  convex immersed hypersurface,  $\Sigma_0$ , which is a graph over  $\Omega$  satisfying condition (i). Since the uniform limit of a sequence of viscosity solutions is also a viscosity solution,  $\Sigma_0$  also satisfies condition (ii) over its interior in the viscosity sense. Moreover, by compactness, we may suppose that there exists  $(p_0, \mathbf{N}_0)$  towards which  $(p_m, \mathbf{N}_m)_{m \in \mathbb{N}}$  converges. Let  $\mathbf{N}'_0$  be the supporting normal to  $\Sigma_0$  at  $p$ . By Proposition 3.9,  $\mathbf{N}_0$  lies on the shortest geodesic joining  $\mathbf{N}_u(p_0)$  to  $\mathbf{N}'_0$ . By concavity of  $K$  (Proposition 2.1, (ii)), and Proposition 5.1, there exists  $\delta > 0$  such that if  $(\lambda_1, \dots, \lambda_{n-1})$  are the principal curvatures of  $\Gamma$  at  $p_0$  with respect to the normal  $\mathbf{N}_0$ , then:

$$K_\infty(\lambda_1, \dots, \lambda_{n-1}) \geq \kappa(p) + \delta.$$

However, by continuity:

$$K_\infty(\lambda_1, \dots, \lambda_{n-1}) = \kappa(p).$$

This is absurd and the result follows.  $\square$

## 6 - Second Order Estimates Along The Boundary.

Let  $M := M^{n+1}$  be an  $(n+1)$ -dimensional Riemannian manifold. Let  $K$  be an admissible convex curvature function which is regular at infinity. In this section we will obtain a-priori estimates along the boundary for the norm of the second fundamental forms of LSC immersed hypersurfaces of prescribed  $K$ -curvature. Explicitly, let  $\kappa \in C^\infty(M)$  be a smooth, strictly positive function. Let  $\Sigma_l := (\Sigma_l, \partial\Sigma_l)$  and  $\Sigma_u = (\Sigma_u, \partial\Sigma_u)$  be LSC immersed hypersurfaces such that:

- (i)  $\Sigma_u > \Sigma_l$ ; and
- (ii)  $K(\Sigma_u) > \kappa > K(\Sigma_l)$ .

As in Section 4, we will denote by  $\mathcal{B}$  the family of all quantities which depend continuously upon the data, being, in this case,  $M$ ,  $K$ ,  $\kappa$ ,  $\Sigma_l$ ,  $\Sigma_u$ , the  $C^1$  jet of  $\Sigma$  and the modulus of continuity,  $m$ , of  $\Sigma$  along the boundary (see Proposition 3.9). We prove:

### Proposition 6.1

There exists  $B > 0$  in  $\mathcal{B}$  such that if  $\Sigma := (\Sigma, \partial\Sigma)$  is a smooth LSC immersed hypersurface in  $M$  such that:

- (i)  $\Sigma_l < \Sigma < \Sigma_u$ ; and
- (ii)  $K(\Sigma) = \kappa \circ i$ ;

and if  $A$  is the shape operator of  $\Sigma$ , then, for all  $p \in \partial\Sigma$ :

$$\|A(p)\| \leq B.$$

For  $p \in \Sigma$ , let  $e_1, \dots, e_n$  be an orthonormal basis diagonalising  $DK_A$  and let  $\mu_1, \dots, \mu_n$  be its corresponding eigenvalues. We define the operator  $\Delta^K$  on functions over  $\Sigma$  by:

$$\Delta^K f = \sum_{i=1}^n \mu_i \text{Hess}^\Sigma(f)_{ii},$$

where  $\text{Hess}^\Sigma$  is the Hessian of the Levi-Civita covariant derivative of  $\Sigma$ . This section is based on the following result which gives a general construction of barrier functions in the non-linear setting, and will also be of use in the sequel:

**Lemma 6.2**

Let  $\varphi : M \rightarrow \mathbb{R}$  be a smooth function such that:

- (i)  $\|\nabla\varphi\| = 1$ ; and
- (ii) the level sets of  $\varphi$  are strictly convex with  $K$ -curvature greater than  $\kappa$ .

Then the restriction of  $\varphi$  to  $\Sigma$  satisfies:

$$\Delta^K \varphi \geq -\|\text{Hess}(\varphi)\| \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

**Proof:** Choose  $q \in \Sigma$ . We construct two orthonormal bases for  $T_q M$ . Let  $L_q$  be the level set of  $\varphi$  passing through  $q$ . Suppose first that  $\nabla\varphi$  and  $N$  are not colinear at  $q$ . Then  $L_q$  and  $\Sigma$  meet transversally at this point. Let  $f_1, \dots, f_{n-1}$  be an orthonormal basis of  $T_q L_q \cap T_q \Sigma$  and complete this to an orthonormal basis  $f_1, \dots, f_n$  of  $T_q \Sigma$ . For  $1 \leq i \leq n-1$ , denote  $f'_i = f_i$ , and complete  $f'_1, \dots, f'_{n-1}$  to an orthonormal basis  $f'_1, \dots, f'_{n+1}$  of  $T_q M$  such that:

- (i)  $f'_n$  is tangent to  $L_q$ ;
- (ii)  $f'_{n+1}$  is normal to  $L_q$ ; and
- (iii)  $f'_n$  makes an angle of at most  $\pi/2$  with  $f_n$ .

Let  $\theta \in ]0, \pi/2]$  be the angle between  $f_n$  and  $f'_n$ . Then:

$$f_n = \cos(\theta) f'_n \pm \sin(\theta) f'_{n+1}.$$

Let  $m_{ij}$  and  $m'_{ij}$  be the matrices of the restrictions of  $\text{Hess}(d_H)$  to  $T_q N$  and  $T_q L_q$  respectively with respect to these bases. Since  $\|\nabla\varphi\| = 1$  and  $f'_{n+1} = \pm \nabla\varphi$ :

$$\text{Hess}(\varphi)(f'_{n+1}, \cdot) = 0.$$

Consequently:

$$(m_{ij}) = \begin{pmatrix} (m'_{ij}) & \cos(\theta)(m'_{in}) \\ \cos(\theta)(m'_{ni}) & \cos^2(\theta)m'_{nn} \end{pmatrix},$$

and so:

$$(m_{ij}) = \cos(\theta)(m'_{ij}) + (1 - \cos(\theta)) \begin{pmatrix} (m'_{ij}) & 0 \\ 0 & m'_{nn} \end{pmatrix} - \sin^2(\theta) \begin{pmatrix} 0 & 0 \\ 0 & m'_{nn} \end{pmatrix}.$$

Consequently, since  $(m'_{ij})$  is positive definite:

$$(m_{ij}) \geq \cos(\theta)(m'_{ij}) - \sin^2(\theta) \begin{pmatrix} 0 & 0 \\ 0 & m'_{nn} \end{pmatrix}.$$

Let  $B^{ij}$  be the matrix of  $DK_A$  with respect to  $f_1, \dots, f_n$ , then, since  $B^{ij}$  is positive definite:

$$\sum_{i,j=1}^n B^{ij} m_{ij} \geq \cos(\theta) \sum_{i,j=1}^n B^{ij} m'_{ij} - \sin^2(\theta) B^{nn} m'_{nn}.$$

However, by concavity of  $K$ , (Axiom (vi)):

$$\begin{aligned} \sum_{i,j=1}^n B^{ij} (m'_{ij} - A_{ij}) &\geq K(m'_{ij}) - K(A_{ij}) \\ \Rightarrow \sum_{i,j=1}^n B^{ij} m'_{ij} &\geq K(m'_{ij}) + \sum_{i,j=1}^n B^{ij} A_{ij} - \kappa. \end{aligned}$$

Thus, by Proposition 2.3, (iv):

$$\begin{aligned} \sum_{i,j=1}^n B^{ij} m'_{ij} &\geq K(m'_{ij}) \\ &> \kappa(p) \\ \Rightarrow \sum_{i,j=1}^n B^{ij} m_{ij} &> \cos(\theta) \kappa(p) - \sin^2(\theta) B^{nn} m'_{nn}. \end{aligned}$$

However:

$$\begin{aligned} \text{Hess}^\Sigma(\varphi) &= \text{Hess}(\varphi) - \langle \nabla \varphi, \mathbf{N} \rangle A \\ \Rightarrow \Delta^K \varphi &\geq \cos(\theta) \kappa(p) - \cos(\theta) \sum_{i=1}^n \mu_i \lambda_i - \sin^2(\theta) B^{nn} m'_{nn} \\ &= -\sin^2(\theta) B^{nn} m'_{nn}. \end{aligned}$$

Finally, by definition,  $\sin(\theta) f_n$  is the orthogonal projection onto  $T\Sigma$  of  $\pm \nabla \varphi$ , and so:

$$|m'_{nn} B^{nn} \sin^2(\theta)| \leq \|\text{Hess}(\varphi)\| \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

The case where  $\nabla \varphi$  and  $\mathbf{N}$  are colinear follows directly from the concavity of  $K$ , and this completes the proof.  $\square$

We may modify this result slightly in different ways. We define the operator  $D_\varphi$  over  $\Sigma$  by:

$$D_\varphi f = \sum_{i=1}^n \mu_i \varphi_{;i} f_{;i}.$$

### Corollary 6.3

Let  $\varphi$  be as in Lemma 6.2. There exists  $\delta, C > 0$  in  $\mathcal{B}(\varphi)$  such that:

$$(\Delta^K + CD_\varphi)\varphi \geq \delta \sum_{i=1}^n \mu_i \text{Hess}(\varphi)_{ii}.$$

**Proof:** In the proof of Lemma 6.2, for  $\delta > 0$  sufficiently small:

$$\sum_{i,j=1}^n (1-\delta) B^{ij} m'_{ij} > \kappa(p).$$

Thus:

$$\Delta^K \varphi \geq \delta \sum_{i,j=1}^n B^{ij} \text{Hess}(\varphi)_{ij} - (1-\delta) \|\text{Hess}(\varphi)\| \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

The result now follows for  $C \geq (1-\delta) \|\text{Hess}(\varphi)\|$ .  $\square$

### Corollary 6.4

Let  $\varphi$  be as in Lemma 6.2. Suppose that  $\varphi \geq \delta > 0$ . There exists  $\epsilon, \alpha, C$  in  $\mathcal{B}(\varphi)$  such that:

$$(\Delta^K + CD_\varphi) \varphi^{1+\alpha} \geq \epsilon \sum_{i=1}^n \mu_i.$$

**Proof:** In the proof of Lemma 6.2, for  $\epsilon_1, \alpha$  sufficiently small:

$$\sum_{i,j=1}^n B^{ij} (1-\epsilon_1) \text{Hess}(\varphi^{1+\alpha})_{ij} > \cos(\theta) \kappa(p) - \sin^2(\theta) B^{nn} \text{Hess}(\varphi^{1+\alpha})(f'_n, f'_n).$$

Thus, reasoning as before:

$$\Delta^K \varphi^{1+\alpha} \geq \epsilon_1 \sum_{i=1}^n \mu_i \text{Hess}(\varphi^{1+\alpha})_{ii} - \|\text{Hess}(\varphi^{1+\alpha})\| \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

However, there exists  $\epsilon > 0$  such that:

$$\epsilon_1 \sum_{i=1}^n \mu_i \text{Hess}(\varphi^{1+\alpha})_{ii} \geq \epsilon \sum_{i=1}^n \mu_i.$$

Finally:

$$D_\varphi \varphi^{1+\alpha} = (1+\alpha) \varphi^\alpha \sum_{i=1}^n \mu_i \varphi_{;i} \varphi_{;i}.$$

Thus, for  $(1+\alpha)\delta^\alpha C \geq \|\text{Hess}(\varphi^{1+\alpha})\|$ , the result follows.  $\square$

Here and in the sequel we will also require the following straightforward relations:

### Proposition 6.5

(i) For all  $p$ :

$$\sum_{i=1}^n \mu_i A_{ii;p} = \kappa_{;p}.$$

(ii) For all  $p, q$ :

$$\sum_{i=1}^n \mu_i A_{ii;pq} = -(D^2 K)^{ij,mn} A_{ij;p} A_{mn;q} + \kappa_{;pq}.$$

**Proof:** This follows by differentiating the equation  $K(A) = \kappa$ .  $\square$

**Proposition 6.6**

Let  $f$  be the signed distance function to  $\Sigma$  and let  $\nu := (n+1)$  denote the outward pointing normal direction to  $\Sigma$ .

(i) Along  $\Sigma$ , for all  $1 \leq i, j \leq n$ :

$$f_{;ij} = A_{ij}, \quad f_{;i\nu} = f_{;\nu i} = f_{;\nu\nu} = 0,$$

where  $A$  is the shape operator of  $\Sigma$ ; and

(ii) along  $\Sigma$ , for all  $1 \leq i, j, k \leq n$ :

$$f_{;ijk} = (\nabla^\Sigma A)_{ijk}, \quad f_{;\nu ij} = -A_{ij}^2,$$

where  $\nabla^\Sigma$  is Levi-Civita covariant derivative of  $\Sigma$ .

**Proof:** See the proof of Lemma 3.16 of [24].  $\square$

Denote  $\Gamma := \partial\Sigma_u = \partial\Sigma_l$ . Choose  $p_0 \in \Gamma$ . The barrier function is constructed from three components. The first is constructed using vector fields as follows: let  $X$  be a vector field defined over a neighbourhood of  $p_0$  in  $M$ . Let  $f : M \rightarrow \mathbb{R}$  be the signed distance function to  $\Sigma$  and define the function  $\varphi_X$  by:

$$\varphi_X = \langle X, \nabla f \rangle.$$

**Proposition 6.7**

The restriction of  $\phi_X$  to  $\Sigma$  satisfies:

$$\Delta^K \varphi_X = O(1)(1 + \sum_{i=1}^n \mu_i) - \varphi_X \sum_{i=1}^n \mu_i \lambda_i^2,$$

where  $\lambda_1 \geq \dots \geq \lambda_n$  and  $\mu_1 \leq \dots \leq \mu_n$  are the eigenvalues of  $A$  and  $DK_A$  respectively, and  $O(1)$  represents terms controlled by  $B$ , for some  $B \in \mathcal{B}(X, p_0)$ .

**Proof:** Choose  $p \in \Sigma$  and let  $e_1, \dots, e_n$  be an orthonormal basis of  $T\Sigma$  with respect to which  $A$  and  $DF_A$  are diagonalised. Let  $\lambda_1 \geq \dots \geq \lambda_n$  and  $\mu_1 \leq \dots \leq \mu_n$  be the corresponding eigenvalues of  $A$  and  $DF_A$  respectively and denote  $\mu = \sum_{i=1}^n \mu_i$ . We extend  $e_1, \dots, e_n$  to an orthonormal basis of  $M$  at  $p$  by defining  $e_{n+1} = \mathbf{N}$ . In the sequel,  $\nu := (n+1)$  denotes the outward pointing normal direction.

Let  $;$  denote covariant differentiation with respect to the Levi-Civita covariant derivative of  $M$ . Let  $R$  be the Riemann curvature tensor of  $M$ . By Propositions 6.5 and 6.6, for all  $1 \leq j \leq n$ :

$$\sum_{i=1}^n \mu_i f_{;jii} = \sum_{i=1}^n \mu_i (f_{;iij} + R_{ij\nu i}) = \kappa_{;j} + O(\mu) = O(1) + O(\mu).$$

Likewise, by Proposition 6.6:

$$f_{;\nu ij} = -A_{ij}^2.$$

Thus:

$$\sum_{i=1}^n \mu_i f_{;\nu ii} = - \sum_{i=1}^n \mu_i \lambda_i^2.$$

By Proposition 2.3, (iv):

$$\sum_{i=1}^n \mu_i \lambda_i = \kappa.$$

In particular, for all  $i$ ,  $\lambda_i \mu_i \leq 1$ . Thus, recalling that  $\|\nabla f\| = 1$  and that  $f_{;ij} = \delta_{ij} \lambda_i$ :

$$\begin{aligned} \sum_{i=1}^n \mu_i \text{Hess}(\varphi_X)_{ii} &= \sum_{i=1}^n \mu_i (X^j_{ii} f_{;j} + 2X^j_i f_{;ji} + X^j f_{;jii}) \\ &= O(1)(1 + \mu) - X^\nu \sum_{i=1}^n \mu_i \lambda_i^2 \\ &= O(1)(1 + \mu) - \varphi_X \sum_{i=1}^n \mu_i \lambda_i^2. \end{aligned}$$

Finally, recall that, for any function  $h$ :

$$\text{Hess}^\Sigma(h) = \text{Hess}(h) - \langle \mathbf{N}, \nabla h \rangle A.$$

Moreover:

$$\langle \mathbf{N}, \nabla \varphi_X \rangle = X^k_{;\nu} f_{;k} + X^k f_{;k\nu}.$$

By Proposition 6.6, (i), the second term on the right hand side vanishes along  $\Sigma$ , and so, using Proposition 2.3, (iv) again:

$$\sum_{i=1}^n \langle \mathbf{N}, \nabla \varphi_X \rangle \mu_i \lambda_i = O(1).$$

Thus:

$$\Delta^K \varphi_X = \sum_{i=1}^n \mu_i \text{Hess}^\Sigma(\varphi_X)_{ii} = O(1)(1 + \sum_{i=1}^n \mu_i) - \varphi_X \sum_{i=1}^n \mu_i \lambda_i^2.$$

This completes the proof.  $\square$

We remove the last term on the right hand side as follows: let  $UM$  be the unitary bundle over  $M$ . Observe first that, since  $\Sigma_l$  and  $\Sigma_u$  are both LSC,  $\mathbf{N}_{\Sigma_l}(p) \neq -\mathbf{N}_{\Sigma_u}(p)$ . There therefore exists a unique geodesic in  $U_p M$  joining  $\mathbf{N}_{\Sigma_l}(p)$  to  $-\mathbf{N}_{\Sigma_u}(p)$ . Let  $\mathbf{N}_0$  be a vector lying on this geodesic, and denote by  $C(p)$  the unique shortest geodesic in  $U_p M$  joining  $\mathbf{N}_0$  to  $\mathbf{N}_{\Sigma_u}(p)$ , and suppose that  $\mathbf{N}_\Sigma(p)$  lies on  $C(p)$ . As in Section 3, for  $q$  near  $p$ , let  $\tau_{p,q}$  be parallel transport from  $q$  to  $p$  along the shortest geodesic from  $q$  to  $p$  and let  $m_0 : [0, \infty[ \rightarrow [0, \infty[$  be a continuous function such that:

- (i)  $\delta_0(0) = 0$ ; and
- (ii)  $D(C(p), \tau_{p,q} \mathbf{N}_\Sigma(q)) \leq m_0(d(p, q))$  for all  $q \in \Sigma$ .



Henceforth, denote  $N_u := N_{\Sigma_u}(p)$ . Suppose first that  $N_0$  and  $N_u$  make an angle of at least  $\pi/2$  with each other at  $p$ . The case where  $N_0$  and  $N_u$  make an acute angle at  $p$  is similar, though slightly simpler, and will be briefly discussed towards the end of this section. Choose  $\delta_0 > 0$  small and let  $V$  be a vector field defined in a neighbourhood of  $p$  such that:

- (i)  $V(p) \in C(p)$ ;
- (ii)  $V(p)$  makes an angle of exactly  $\pi/2 - \delta_0$  with  $N_0$  at  $p$ ; and
- (iii)  $(\nabla V)(p) = 0$ .

*Remark:* Observe that, for  $\delta_0$  sufficiently small,  $V(p)$  also makes an angle of strictly less than  $\pi/2$  with  $N_u$ .  $\square$

The entire construction is illustrated in Figure 2. Observe that, throughout much of the rest of this section, the data necessarily incorporates  $(V, N_0, m_0, \delta_0)$ . The correct choice of  $(N_0, m_0)$  will be made presently.

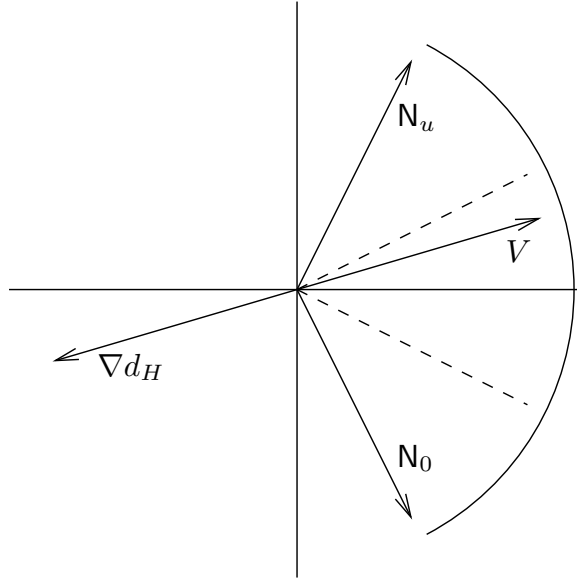


Figure 2

Observe that, by Conditions (i) and (ii), there exists  $r_0, \epsilon_0 > 0$  in  $\mathcal{B}(p_0, V, N_0, m_0, \delta_0)$  such that, throughout  $\Sigma \cap B_{r_0}(p)$ ,  $\varphi_V \geq \epsilon_0$ . We thus define the first order operator  $\mathcal{D}_1$  such that, for any function  $h$ :

$$\mathcal{D}_1 h = \frac{2}{\varphi_V} \sum_{i=1}^n \mu_i \varphi_{V;i} h_{;i}.$$

We define the operator  $\mathcal{L}_1$  by:

$$\mathcal{L}_1 = \Delta^K + \mathcal{D}_1.$$

**Proposition 6.8**

Using the notation of Proposition 6.7, the restriction of  $\varphi_X \varphi_V^{-1}$  to  $\Sigma \cap B_{r_0}(p_0)$  satisfies:

$$\mathcal{L}_1(\varphi_X \varphi_V^{-1}) = O(1)(1 + \sum_{i=1}^n \mu_i),$$

where  $O(1)$  represents terms controlled by  $B$ , for some  $B \in \mathcal{B}(p_0, X, V, \mathbf{N}_0, m_0, \delta_0)$ .

**Proof:** We use the notation of the proof of Proposition 6.7. By the product rule and the chain rule:

$$\begin{aligned} \Delta^K(\varphi_X \varphi_V^{-1}) &= \sum_{i=1}^n \mu_i (\varphi_V^{-1} \text{Hess}^\Sigma(\varphi_X)_{ii} + 2(\nabla^\Sigma \varphi_X)_i (\nabla^\Sigma \varphi_V^{-1})_i + \varphi_X \text{Hess}^\Sigma(\varphi_V^{-1})_{ii}) \\ &= \sum_{i=1}^n \mu_i (\varphi_V^{-1} \text{Hess}^\Sigma(\varphi_X)_{ii} - 2\varphi_V^{-2} (\nabla^\Sigma \varphi_X)_i (\nabla^\Sigma \varphi_V)_i \\ &\quad - \varphi_X \varphi_V^{-2} \text{Hess}^\Sigma(\varphi_V)_{ii} + 2\varphi_X \varphi_V^{-3} (\nabla^\Sigma \varphi_V)_i (\nabla^\Sigma \varphi_V)_i) \\ &= \varphi_V^{-1} (\Delta^K \varphi_X) - 2\varphi_V^{-2} \sum_{i=1}^n \mu_i (\varphi_X)_{;i} (\varphi_V)_{;i} \\ &\quad + 2\varphi_X \varphi_V^{-3} \sum_{i=1}^n \mu_i (\varphi_V)_{;i} (\varphi_V)_{;i} - \varphi_X \varphi_V^{-2} (\Delta^K \varphi_V). \end{aligned}$$

Thus, by Proposition 6.7, bearing in mind that  $|\varphi_X|, |\varphi_V|, |\varphi_V^{-1}| = O(1)$ :

$$\begin{aligned} \Delta^K(\varphi_X \varphi_V^{-1}) &= O(1)(1 + \mu) - 2\varphi_V^{-1} \sum_{i=1}^n \mu_i \text{Ln}(\varphi_V)_{;i} (\varphi_X)_{;i} \\ &\quad - 2\varphi_X \sum_{i=1}^n \mu_i \text{Ln}(\varphi_V)_{;i} (\varphi_V^{-1})_{;i} \\ &= O(1)(1 + \mu) - 2 \sum_{i=1}^n \mu_i \text{Ln}(\varphi_V)_{;i} (\varphi_X \varphi_V^{-1})_{;i} \\ &= O(1)(1 + \mu) - \mathcal{D}_1(\varphi_X \varphi_V^{-1}). \end{aligned}$$

This completes the proof.  $\square$

We now construct the second component of the barrier function. This will be a function whose restriction to  $\Sigma$  is non-negative but subharmonic. Let  $P \subseteq M$  be the geodesic hyperplane passing through  $p$  whose normal at  $p$  is equal to  $\mathbf{N}_0$ . In other words:

$$P = \{\text{Exp}(X) \text{ s.t. } X \in T_p M, \text{ and } \langle X, \mathbf{N}_0(p) \rangle = 0\}.$$

Let  $A_P$  be the shape operator of  $P \cap \Sigma_u$ . Thus, if  $X$  and  $Y$  are vector fields in  $P \cap \Sigma_u$  and if  $N$  is a normal vector to  $P \cap \Sigma_u$ :

$$A_P(N)(X, Y) = -\langle \nabla_X Y, N \rangle.$$

**Proposition 6.9**

For  $\delta_0$  sufficiently small, at  $p$ :

$$K_\infty(A_P(V)) > \kappa.$$

**Proof:** Suppose first that  $\delta_0 = 0$ . Let  $A_u$  be the shape operator of  $\Sigma_u$  at  $p_0$ . Since  $P$  is totally geodesic at  $p_0$ , elementary linear algebra yields, at  $p_0$ :

$$\begin{aligned} A_P(V) &= \frac{1}{\langle V, \mathbf{N}_u(p) \rangle} A_P(\mathbf{N}_u) \\ &= \frac{1}{\langle V, \mathbf{N}_u(p) \rangle} A_u(p)|_{TP \cap \Sigma_u} \\ &\geq A_u(p)|_{TP \cap \Sigma_u}. \end{aligned}$$

Thus:

$$\begin{aligned} K_\infty(A_P(V)) &\geq K_\infty(A_u(p)|_{TP \cap \Sigma_u}) \\ &> K(A_u(p)) \\ &> \kappa(p). \end{aligned}$$

Since this relation is preserved by small perturbations of  $\delta_0$ , the result follows.  $\square$

Choose  $\delta_1 > 0$ . Bearing in mind Proposition 6.9, we define a strictly concave embedded hypersurface  $H$  in a neighbourhood of  $p$  such that:

- (i)  $H$  passes through  $p$ ;
  - (ii) the outward pointing normal to  $H$  at  $p$  is  $-V(p)$ ; and
- if  $A_H$  is the shape operator of  $H$  at  $p$ , then:
- (iii) bearing in mind that  $A_H$  is negative definite:

$$A_V(p) = \delta_1 \text{Id} - A_H|_{TP_V \cap \Sigma_u}; \text{ and}$$

- (iv)  $K(A_H)(p) > \kappa(p)$ .

Let  $d_H$  be the signed distance to  $H$  in  $M$ . We first show subharmonicity of  $d_H$ . Non-negativity will be proven later, as it depends on an appropriate choice of  $(\mathbf{N}_0, m_0)$ . For  $C > 0$ , define the first order operator  $\mathcal{D}_2$  such that, for all  $h$ :

$$\mathcal{D}_2 h = -C \sum_{i=1}^n \mu_i d_{H;i} h_{;i}.$$

We now define  $\mathcal{L}_2$  by:

$$\mathcal{L}_2 = \Delta^K + \mathcal{D}_1 + \mathcal{D}_2.$$

### Proposition 6.10

Using the notation of Proposition 6.8, there exists  $C \in \mathcal{B}(p_0, X, V, \mathbf{N}_0, m_0, \delta_0, H)$  such that, after reducing  $r_0$  if necessary, throughout  $\Sigma \cap B_{r_0}(p_0)$ :

$$\mathcal{L}_2 d_H \leq 0.$$

**Proof:** For  $\delta_a > 0$  sufficiently small, bearing in mind that  $-d_H$  is convex, by Corollary 6.3, for sufficiently large  $C$ :

$$(\Delta^K + \mathcal{D}_2)d_H \leq \delta_a \sum_{i=1}^n \mu_i \text{Hess}(d_H)_{ii}.$$

Since  $\nabla d_H$  is never tangent to  $\Sigma$ , there exists  $\delta_b > 0$  such that:

$$\text{Hess}(d_H)_{ii}|_{T\Sigma} \leq -\delta_b \text{Id}.$$

Thus, denoting  $\delta_c = \delta_a \delta_b$ :

$$(\Delta^K + \mathcal{D}_2)d_H \leq -\delta_c \sum_{i=1}^n \mu_i.$$

It remains to consider the contribution from  $\mathcal{D}_1$ . As in the proof of Proposition 6.7:

$$(\varphi_V)_{;i} = V^k_{;i} f_k + V^k f_{;ki} = V^\nu_{;i} + \lambda_i V^i.$$

By Property (iii) of  $V$ , after reducing  $r_0$  if necessary, we may assume that:

$$\left| 2 \sum_{i=1}^n \mu_i \varphi_V^{-1} V^\nu_{;i} d_{H;i} \right| \leq \frac{\delta_c}{2} \mu.$$

Moreover, since  $(\nabla d_H + V)(p) = 0$  at  $p$ , after reducing  $r_0$  further if necessary, we may assume that:

$$\|\nabla d_H + V\| \leq \frac{\varphi_V}{2\kappa\|V\|} \frac{\delta_c \mu}{2}.$$

Consequently, bearing in mind Proposition 2.3, (iv):

$$\begin{aligned} 2\varphi_V^{-1} \sum_{i=1}^n \mu_i \lambda_i V^i d_{H;i} &= -2\varphi_V^{-1} \sum_{i=1}^n \mu_i \lambda_i V^i V^i \\ &\quad + 2\varphi_V^{-1} \sum_{i=1}^n \mu_i \lambda_i V^i (d_{H;i} + V^i) \\ &\leq \frac{\delta_c \mu}{2}. \end{aligned}$$

Combining these relations yields:

$$\mathcal{L}_2 d_H \leq 0,$$

and this completes the proof.  $\square$

We now verify that the addition of the term  $\mathcal{D}_2$  does not affect the conclusion of Proposition 6.8. Indeed:

### Proposition 6.11

Using the notation of Proposition 6.8, for all  $C > 0$ , the restriction of  $(\varphi_X \varphi_V^{-1})$  to  $\Sigma \cap B_{r_0}(p_0)$  satisfies:

$$\mathcal{D}_2(\varphi_X \varphi_V^{-1}) = O(1) \left( 1 + \sum_{i=1}^n \mu_i \right).$$

**Proof:** For any vector field,  $Y$ :

$$\phi_{Y;i} = Y^\nu_{;i} + Y_i \lambda_i.$$

The result now follows by Proposition 2.3, (iv).  $\square$

### Corollary 6.12

Using the notation of Proposition 6.8, for all  $C > 0$ , the restriction of  $(\varphi_X \varphi_V^{-1})$  to  $\Sigma \cap B_{r_0}(p_0)$  satisfies:

$$\mathcal{L}_2(\varphi_X \varphi_V^{-1}) = O(1)(1 + \sum_{i=1}^n \mu_i).$$

The third component of the barrier function is simply the squared distance to  $p$ . Let  $d_0$  denote the distance to  $p_0$  in  $M$ :

### Proposition 6.13

There exists  $\epsilon_1 > 0$  in  $\mathcal{B}(p_0, V, N_0, m_0, \delta_0, H)$  such that, after reducing  $r_0$  if necessary, throughout  $\Sigma \cap B_{r_0}(p_0)$ :

$$\mathcal{L}_2 d_0^2 \geq \epsilon_1 (1 + \sum_{i=1}^n \mu_i).$$

**Proof:** We continue to use the notation of the proof of Proposition 6.7. Since  $\mu \geq 1$ , by Proposition 2.3, (iv):

$$\Delta^K(d_0^2) \geq 4\epsilon_1(1 + \mu) - 2d_P \langle d_P, N \rangle \kappa.$$

After reducing  $r$  if necessary, we may assume that, throughout  $B_{r_0}(p_0)$ :

$$2d_0 \langle d_0, N \rangle \kappa < \epsilon,$$

and so, throughout  $B_{r_0}(p_0)$ :

$$\Delta^K(d_0^2) \geq 3\epsilon(1 + \mu).$$

As in the proof of Proposition 6.10, reducing  $r_0$  further if necessary:

$$\left| 4d_0 \sum_{i=1}^n \mu_i \varphi_V^{-1} V^\nu_{;i} d_{0;i} \right| \leq \epsilon \mu.$$

Moreover, for all  $i$ , by Proposition 2.3, (iv), bearing in mind that  $\mu_i, \lambda_i \geq 0$ :

$$|4\varphi_V^{-1} \mu_i \lambda_i V^i d_{0;i}| = O(1).$$

Thus, by reducing  $r_0$  even further if necessary:

$$|4d_0 \varphi_V^{-1} \mu_i \lambda_i V^i d_{0;i}| \leq \epsilon_1.$$

Combining these relations yields:

$$|\mathcal{D}_1 d_0^2| \leq \epsilon_1(1 + \mu).$$

In like manner, after reducing  $r_0$  if necessary:

$$|\mathcal{D}_2 d_0^2| \leq \epsilon_1(1 + \mu).$$

Thus:

$$\mathcal{L}_2(d_0^2) \geq \epsilon_1(1 + \mu),$$

and this completes the proof.  $\square$

We have now obtained all the estimates we require on each of the components to construct the barrier functions used to prove Proposition 6.1 in the case where  $\mathbf{N}_u(p)$  makes an angle of at least  $\pi/2$  with  $\mathbf{N}_0(p)$  at  $p$ . When the angle between  $\mathbf{N}_u(p)$  and  $\mathbf{N}_0(p)$  is less than  $\pi/2$ , we merely choose  $V(p) = \mathbf{N}_u(p)$ , and the reader may verify that the conclusions of Corollary 6.12 and Propositions 6.10 and 6.13 continue to hold. We now prove Proposition 6.1:

**Proof of Proposition 6.1:** We assume the contrary and obtain a contradiction. Let  $(\Sigma_n)_{n \in \mathbb{N}} = (i_n, S_n)_{n \in \mathbb{N}}$  be a sequence of smooth, LSC, immersed hypersurfaces and let  $(p_n)_{n \in \mathbb{N}} \in \partial \Sigma_u = \partial \Sigma_l$  be a sequence of points such that, for all  $n$ :

- (i)  $\Sigma_u > \Sigma_n > \Sigma_l$ ;
- (ii)  $K(\Sigma_n) = \kappa \circ i_n$ ;

and, if  $A_n$  is the shape operator of  $\Sigma$ , then:

$$\|A_n(p_n)\|_{n \in \mathbb{N}} \rightarrow +\infty.$$

By Lemma 3.8, we may suppose that there exists a  $C^{0,1}$  LSC immersed hypersurface  $\Sigma_0$  towards which  $(\Sigma_n)_{n \in \mathbb{N}}$  converges uniformly. Likewise, we may assume that there exists  $p_0 \in \partial \Sigma_0$  towards which  $(p_n)_{n \in \mathbb{N}}$  converges. For simplicity, assume that  $p_n = p_0$  for all  $n$ .

Recall that, since  $\Gamma := \partial \Sigma_u = \partial \Sigma_0$  is smooth, the outward pointing normal to  $\Sigma_0$  at  $p_0$  is well defined. Provisionally, let  $\mathbf{N}_0$  be this outward pointing normal.

Choose  $\delta_1 > 0$  and define the geodesic hyperplane,  $P$  and the smooth, embedded hypersurface,  $H$ , as outlined previously. Let  $h : H \rightarrow \mathbb{R}$  be a smooth function such that:

- (i)  $h(p_0) = 0$ ;
- (ii)  $dh(p_0) = 0$ ; and
- (iii)  $\text{Hess}(h)(p_0) < \delta_1 \text{Id}$ .

We claim that, for  $\delta_0$  sufficiently small, after reducing  $r_0$  is necessary, the connected component of  $\Sigma_0 \cap B_{r_0}(p_0)$  containing  $p_0$  lies above the graph of  $h$  over  $H$ . Indeed, denote this connected component by  $\Sigma_{0,p_0}$ . For  $r_0$  sufficiently small,  $\Sigma_{0,p_0}$  lies inside  $\Sigma_u$ . Moreover,

by convexity, and by definition of  $P$ , for  $r_0$  sufficiently small,  $\Sigma_{0,p_0}$  lies below  $P$  (i.e. on the other side of the hypersurface  $P$  from the vector  $V$ ). Thus, if we denote by  $P'$  the connected component of  $P \setminus (P \cap \Sigma_u)$  contained inside  $\Sigma_u$  and if (after extending  $\Sigma_u$  smoothly beyond  $\Gamma := \partial\Sigma_u$ ) we denote by  $\Sigma'_u$  the connected component of  $\Sigma_u \setminus (P \cap \Sigma_u)$  lying below  $P$ , then  $\Sigma_0$  lies in the region bounded by  $P' \cup \Sigma'_u$ . It thus suffices to prove that both  $P'$  and  $\Sigma'_u$  lie above the graph of  $f$  over  $H$ .

Suppose now that  $\delta_0 = 0$ . By hypothesis,  $P \cap \Sigma_u$  lies strictly above the graph of  $h$  over  $H$ . Since this property is preserved by small perturbations, it remains true for small positive values of  $\delta_0$ . Since  $\Sigma_0$  and  $P$  are both transverse to  $H$  at  $p$ , the assertion now follows.

We see, moreover, that this continues to hold for small perturbations of  $N_0$ , and so, perturbing  $N_0$  slightly towards  $N_{\Sigma_l}$  (and thus increasing  $C(p)$  slightly), for all  $n$  sufficiently large,  $d_H$  is non-negative over  $\partial(\Sigma_n \cap B_r(p))$ . In addition by Proposition 3.9, there exists a continuous function  $m_0 : [0, \infty[ \rightarrow [0, \infty[$  such that  $m_0(0) = 0$ , and, for all sufficiently large  $n$  and for all  $q \in \Sigma_n$ :

$$D(C(p), \tau_{p,q} N_{\Sigma_n}(q)) \leq m_0(d(q, p)).$$

Choose  $n$  large and denote  $\Sigma := \Sigma_n$  and  $\varphi = \varphi_X \varphi_V^{-1}$ . By Proposition 6.13, there exists  $A_- > 0$  such that, throughout  $B_{r_0}(p_0) \cap \Sigma$ :

$$\mathcal{L}_2(\varphi - A_- d_0^2) < 0.$$

By construction, there exists  $\delta_a > 0$  such that  $d_H \geq \delta_a d_p^2$  along  $\Gamma_1 = \partial\Sigma \cap B_r(p)$  and  $d_H \geq \delta_a$  along  $\Gamma_2 := \Sigma \cap \partial B_{r_0}(p_0)$ . Bearing in mind Proposition 6.10, there therefore exists  $B_- > 0$  such that:

- (i)  $\mathcal{L}_2(\varphi + B_- d_H - A_- d_0^2) < 0$  throughout  $B_{r_0}(p_0) \cap \Sigma$ ; and
- (ii)  $\varphi \geq A_- d_p^2 - B_- d_H$  along  $\partial(B_{r_0}(p_0) \cap \Sigma)$ .

It thus follows by the maximum principal that, throughout  $B_{r_0}(p_0) \cap \Sigma$ :

$$\varphi \geq A_- d_0^2 - B_- d_H.$$

Likewise, reducing  $r_0$  further if necessary, there exists  $A_+$  and  $B_+$  such that, throughout  $B_{r_0}(p_0) \cap \Sigma$ :

$$\varphi \leq B_+ d_H - A_+ d_0^2.$$

We thus obtain a-priori bounds on  $\nabla^\Sigma \varphi$  at  $P$ . Let  $f$  be the signed distance function to  $\Sigma$ . For all  $Y$ , since  $\varphi_X(p_0) = 0$ :

$$\text{Hess}(f)(X, Y) = \langle \nabla \varphi, Y \rangle \varphi_V(p_0) - \langle \nabla_Y X, N \rangle.$$

Thus, since  $X$  is arbitrary, we obtain a-priori bounds on  $\text{Hess}(f)(X, Y)$  for all pairs of vectors  $X, Y \in T_P \Sigma$  where at least one of  $X$  or  $Y$  is tangent to  $\partial\Sigma$ . Since the second

fundamental form of  $\Sigma$  is the restriction to  $T\Sigma$  of the hessian of  $f$ , and, recalling that  $\Sigma = \Sigma_n$ , we deduce that there exists  $B$  such that:

$$\|A_n(X, Y)\|(p_n) \leq B\|X\|\|Y\|,$$

for all  $n$  and for all such pairs of vectors. However, by hypotheses,  $\|A_n(p_0)\| \rightarrow +\infty$ , and it follows that  $\|A_n(X_n, X_n)\| \rightarrow +\infty$  where, for all  $n$ ,  $X_n$  is the unit vector normal to  $\partial\Sigma$  in  $T\Sigma$ .

However, we may assume that  $(X_n)_{n \in \mathbb{N}}$  converges to  $X_0$ , say, which is normal to  $\Gamma$  at  $p_0$ . Let  $\lambda'_1 \leq \dots \leq \lambda'_{n-1}$  be the eigenvalues of  $A_\Gamma(X_0)$ . For all  $m$ , let  $\lambda_{1,m} \leq \dots \leq \lambda_{n,m}$  be the eigenvalues of  $A_n$ . By definition,  $(\lambda_{n,m})_{m \in \mathbb{N}} \rightarrow +\infty$ . By Lemma 1.2 of [4] and the bounds already obtained, for all  $1 \leq i \leq n-1$ :

$$(\lambda_{i,m})_{m \in \mathbb{N}} \rightarrow \lambda'_i.$$

Suppose first that  $K$  satisfies Axiom (vii'). By concavity,  $K(x_1, \dots, x_{n-1}, t)$  converges to  $K_\infty(x_1, \dots, x_{n-1})$  locally uniformly in  $(x_1, \dots, x_{n-1})$  as  $t \rightarrow +\infty$ . Thus, by Proposition 4.1:

$$\begin{aligned} \lim_{m \rightarrow +\infty} K(\lambda_{1,m}, \dots, \lambda_{n,m}) &= K_\infty(\lambda'_1, \dots, \lambda'_{n-1}) \\ &\geq \kappa(p_0) + \delta \\ &> \kappa(p_0), \end{aligned}$$

which is absurd. Suppose now that  $K$  satisfies Axiom (vii), then, in the same manner, we obtain:

$$\lim_{m \rightarrow +\infty} K(\lambda_{1,m}, \dots, \lambda_{n,m}) = +\infty > \kappa(p),$$

which is likewise absurd. There thus exists  $B_2 \geq 0$  such that, for all  $n$ :

$$\|A_n(p_n)\| \leq B_2,$$

which is absurd, and this completes the proof.  $\square$

## 7 - Second Order Bounds Over the Interior.

Let  $M := M^{n+1}$  be a Hadamard manifold of sectional curvature bounded above by  $-1$ . Let  $K$  be an admissible convex curvature function. In this section, we obtain a-priori bounds for the norms of the second fundamental forms of locally convex immersed hypersurfaces of prescribed  $K$ -curvature. Explicitly, let  $\kappa \in C^\infty(M)$  be a smooth, strictly positive function. Let  $\Sigma = (\Sigma^n, \partial\Sigma^n)$  be a smooth LSC immersed hypersurface in  $M$  such that:

$$K(\Sigma) = \kappa.$$

As in Section 6, let  $\mathcal{B}$  denote the family of all quantities which depend continuously upon the data, being, in this case,  $M$ ,  $K$ ,  $\kappa$ , and the  $C^1$  jet of  $\Sigma$ . Likewise, for any quantity,  $X$ , we define  $\mathcal{B}(X)$  as before. We obtain the following complementary results:

### Proposition 7.1

Suppose that  $0 < \kappa < 1$  and that the sectional curvature of  $M$  is bounded above by  $-1$ . Then there exists  $B > 0$  in  $\mathcal{B}$  such that if  $A$  is the shape operator of  $\Sigma$ , then, for all  $p \in \Sigma$ :

$$\|A(p)\| \leq \text{Max}(B, \sup_{q \in \partial\Sigma} \|A(q)\|).$$



**Proposition 7.2**

Choose  $p_0 \in M$  and  $R > 0$ . There exists  $B > 0$  in  $\mathcal{B}(p, R)$  such that if:

(i)  $\Sigma \subseteq B_R(p)$ ; and

(ii)  $\kappa < \frac{1}{R}\mu_\infty(K)$ ,

and if  $A$  is the shape operator of  $\Sigma$ , then, for all  $p \in \Sigma$ :

$$\|A(p)\| \leq \text{Max}(B, \sup_{q \in \partial\Sigma} \|A(q)\|).$$

Let  $N$  and  $A$  be the unit exterior normal vector field and the shape operator of  $\Sigma$  respectively. In the sequel, we raise and lower indices with respect to  $A$ . Thus:

$$A^{ij}A_{jk} = \delta^i_k,$$

where  $\delta$  is the Krönecker delta function. We recall the commutation rules of covariant differentiation in a Riemannian manifold:

**Lemma 7.3**

Let  $R^\Sigma$  and  $R^M$  be the Riemann curvature tensors of  $\Sigma$  and  $M$  respectively. Then:

(i) For all  $i, j, k$ :

$$A_{ij;k} = A_{kj;i} + R_{kij\nu}^M,$$

where  $\nu$  represents the direction normal to  $\Sigma$ ; and

(ii) For all  $i, j, k, l$ :

$$A_{ij;kl} = A_{ij;lk} + R_{kli}^\Sigma A_{pj} + R_{klj}^\Sigma A_{pi}.$$

**Proof:** See Lemma 6.3 of [22].  $\square$

**Corollary 7.4**

For all  $i, j, k$  and  $l$ :

$$A_{ij;kl} = A_{kl;ij} + R_{kj\nu i;l}^M + R_{li\nu k;j}^M + R_{jlk}^\Sigma A_{pi} + R_{jli}^\Sigma A_{pk}.$$

**Proof:** See Corollary 6.4 of [22].  $\square$

Choose  $p \in \Sigma$ . Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $A$  at  $p$ . Choose an orthonormal basis,  $(e_1, \dots, e_n)$  of  $T_p\Sigma$  with respect to which  $A$  is diagonal such that  $a := \lambda_1 = A_{11}$  is the highest eigenvalue of  $A$  at  $p$ . We extend this to a frame in a neighbourhood of  $p$  by parallel transport along geodesics. We likewise extend  $a$  to a function defined in a neighbourhood of  $p$  by:

$$a = A(e_1, e_1).$$

Viewing  $\lambda_1$  also as a function defined near  $p$ ,  $\lambda_1 \geq a$  and  $\lambda_1 = a$  at  $p$ , we recall:

**Proposition 7.5**

For all  $i$ , at  $p$ :

$$a_{;ii} = A_{11;ii}.$$

**Proof:** See Proposition 6.5 of [22].  $\square$

Let  $\mu_1 \leq \dots \leq \mu_n$  be the eigenvalues of  $DK_A$  at  $p$ . We define the Laplacian  $\Delta$  such that, for all functions  $f$ :

$$\Delta f = \sum_{i=1}^n \mu_i f_{;ii}.$$

For  $\psi \in C^\infty(M)$ , define the first order operator  $D_\psi$  such that, for all functions,  $f$ :

$$D_\psi f = \sum_{i=1}^n \mu_i \psi_{;i} f_{;i}.$$

Define  $I, J \subseteq \{1, \dots, n\}$  by:

$$I = \{1 \leq i \leq n \text{ s.t. } \mu_i \leq 4\mu_1\}, \quad J = \{1 \leq i \leq n \text{ s.t. } \mu_i > 4\mu_1\}.$$

### Proposition 7.6

For all  $C \geq 0$ , there exists  $K > 0$  in  $\mathcal{B}(\psi, C)$  such that, if  $a > 1$ , then:

$$(\Delta + C\mathcal{D}_\psi)\text{Log}(a)(P) \geq -K(1 + \sum_{i=1}^n \mu_i) - \frac{9}{8} \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2.$$

**Proof:** In this proof, we denote by  $;$  covariant differentiation with respect to the Levi-Civita covariant derivative of  $\Sigma$ . By Proposition 7.5 and Corollary 7.4:

$$\begin{aligned} a_{;ii} &= A_{11;ii} \\ &= A_{ii;11} + R_{i1\nu 1;i}^M + R_{i1\nu i;1}^M + R_{1ii}^{\Sigma p} A_{p1} + R_{1i1}^{\Sigma p} A_{pi}. \end{aligned}$$

However, at  $p$ , by Proposition 6.5,  $(ii)$ :

$$\sum_{i=1}^n \frac{\mu_i}{\lambda_1} A_{ii;11} = -\frac{1}{\lambda_1} (D^2 K)^{ij,mn} A_{ij;1} A_{mn;1} + \frac{1}{\lambda_1} \kappa_{;11}.$$

Thus, at  $p$ , bearing in mind Proposition 6.5:

$$\begin{aligned} \Delta \text{Log}(a) &= \frac{1}{\lambda_1} \kappa_{;11} - \frac{1}{\lambda_1} (D^2 K)^{ij,mn} A_{ij;1} A_{mn;1} - \sum_{i=1}^n \frac{\mu_i}{\lambda_1^2} A_{11;i} A_{11;i} \\ &\quad + \sum_{i=1}^n \frac{\mu_i}{\lambda_1} (R_{i1\nu 1;i}^M + R_{i1\nu i;1}^M) + \sum_{i,j=1}^n \frac{\mu_i}{\lambda_1} (R_{1ii}^{\Sigma p} A_{p1} + R_{1i1}^{\Sigma p} A_{pi}). \end{aligned}$$

We consider each contribution separately. Since, for all  $a, b \in \mathbb{R}$  and for all  $\eta > 0$ ,  $(a+b)^2 \leq (1+\eta)a^2 + (1+\eta^{-1})b^2$ , by Lemma 7.3,  $(i)$ , for all  $i \in J$ :

$$\frac{9}{8} A_{11;i}^2 = \frac{9}{8} (A_{i1;1} + R_{i1\nu 1}^M)^2 \leq \frac{5}{4} A_{i1;1}^2 + O(1).$$

Thus, by Proposition 2.3, (vi), bearing in mind the definition of  $J$  and the fact that  $\lambda_1 \geq 1$ :

$$\begin{aligned} & -\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} - \frac{9}{8}\sum_{i \in J} \frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i} \\ & \geq \sum_{i \in J} \left( \frac{2(\mu_1 - \mu_i)}{\lambda_1(\lambda_i - \lambda_1)} - \frac{5}{4} \frac{\mu_i}{\lambda_1^2} \right) A_{i1;1}^2 + O(\mu) \\ & \geq \sum_{i \in J} \frac{\mu_1}{(\lambda_1 - \lambda_j)\lambda_1} A_{i1;1}^2 + O(\mu) \\ & \geq O(\mu). \end{aligned}$$

Thus:

$$-\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} - \sum_{i \in J} \frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i} \geq \frac{1}{8}\sum_{i \in J} \frac{\mu_i}{\lambda_1^2}A_{11;i}A_{11;i} + O(\mu).$$

For all  $\xi$ ,  $X$  and  $Y$ :

$$\begin{aligned} \nabla^\Sigma \xi(Y; X) &= \nabla^M \xi(Y; X) - A(X, Y)\xi(N); \text{ and} \\ X\xi(N) &= \nabla^M \xi(N; X) + \xi(AX). \end{aligned}$$

Thus:

$$\begin{aligned} R_{i1\nu 1; i}^M &= (\nabla^M R^M)_{i1\nu 1; i} + \lambda_i(1 - \delta_{i1})R_{1\nu\nu 1}^M + \lambda_i R_{i1i1}^M, \\ R_{i1\nu i; 1}^M &= (\nabla^M R^M)_{i1\nu i; 1} - \lambda_1(1 - \delta_{i1})R_{i\nu\nu i}^M - \lambda_1 R_{i1i1}^M. \end{aligned}$$

Bearing in mind that  $\lambda_1 \geq 1$ , by Proposition 2.3, (iv), there exists  $K_3$ , which only depends on  $M$  such that, if we denote  $\mu = \sum_{i=1}^n \mu_i$ , then:

$$\sum_{i=1}^n \frac{\mu_i}{\lambda_1} (R_{i1\nu 1; i}^M + R_{i1\nu i; 1}^M) \geq -K_3(1 + \mu).$$

Moreover:

$$R_{1ii}^{\Sigma \ p} A_{p1} + R_{1i1}^{\Sigma \ p} A_{pi} = R_{1ii1}^M (\lambda_1 - \lambda_i) + \lambda_1 \lambda_i (\lambda_1 - \lambda_i).$$

Bearing in mind that  $\lambda_1 \geq 1$  and that  $\lambda_1 \geq \lambda_i$  for all  $i$ , there exists  $K_2$ , which only depends on  $M$  such that:

$$\sum_{i,j=1}^n \frac{\mu_i}{\lambda_1} (R_{1ii}^{\Sigma \ p} A_{p1} + R_{1i1}^{\Sigma \ p} A_{pi}) \geq -K_2(1 + \mu).$$

Finally:

$$\begin{aligned} \kappa_{;11} &= \text{Hess}^\Sigma(\kappa)(e_1, e_1) \\ &= \text{Hess}^M(\kappa)(e_1, e_1) - \langle \nabla \kappa, N \rangle A_{11} \\ &= \text{Hess}^M(\kappa)(e_1, e_1) - \lambda_1 d\kappa(N) \end{aligned}$$

Bearing in mind that  $\lambda_1 \geq 1$ , there thus exists  $K_3$ , which only depends on  $M$  and  $\kappa$  such that:

$$\frac{1}{\lambda_1} \kappa_{;11} \geq -K_3.$$

Combining these relations, there exists  $K_4$  such that:

$$\Delta \text{Log}(a) \geq K_4(1 + \mu) - \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2 + \frac{1}{8} \sum_{i \in J} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2.$$

Finally, bearing in mind that  $\lambda_1 = a \geq 1$ :

$$\begin{aligned} C\mathcal{D}_\psi \text{Log}(a) &= C \sum_{i=1}^n \frac{\mu_i}{\lambda_1} A_{11;i} \psi_{;i} \\ &\geq -\frac{1}{8} \sum_{i=1}^n \frac{\mu_i}{\lambda_1^2} A_{11;i}^2 + O(\mu). \end{aligned}$$

The result now follows by combining the above relations.  $\square$

We recall that a function  $f$  is said to satisfy  $\Delta f \geq g$  in the weak sense if and only if, for all  $P \in \Sigma$ , there exists a smooth function  $\varphi$ , defined near  $P$  such that:

- (i)  $f \geq \varphi$  near  $P$ ;
- (ii)  $f = \varphi$  at  $P$ ; and
- (iii)  $\Delta \varphi \geq g$  at  $P$ .

### Corollary 7.7

With the same  $K$  as in Proposition 7.6, if  $\lambda_1 \geq 1$ , then:

$$(\Delta \text{Log} + C\mathcal{D}_u)(\lambda_1) \geq -K(1 + \sum_{i=1}^n \mu_i) - \frac{9}{8} \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2,$$

in the weak sense.

**Proof:** Near  $P \in \Sigma$ ,  $\lambda_1 \geq a$  and  $\lambda_1 = a$  at  $P$ . Since  $P \in \Sigma_0$  is arbitrary, and since  $a$  is smooth at  $P$ , the result follows.  $\square$

Choose  $p_0 \in M$  and define  $\delta$  by:

$$\delta = d(x, p_0).$$

### Proposition 7.8

If the sectional curvature of  $M$  is bounded above by  $-1$ , then there exists  $\epsilon, C > 0$  and  $\alpha \in ]0, 1[$  in  $\mathcal{B}(p_0)$  such that, over  $\Sigma$  and away from  $p_0$ :

$$(\Delta + C\mathcal{D}_\delta) \delta^{1+\alpha} \geq \epsilon(1 + \sum_{i=1}^n \mu_i).$$

**Proof:** Trivially,  $\|\nabla \delta\| = 1$ . Moreover, since the level sets of  $\delta$  are geodesic spheres and since  $M$  is a Hadamard manifold, they are strictly convex. Finally, since the sectional curvature of  $M$  is bounded above by  $-1$ , by Axiom (iii) and (v), the level sets have  $K$ -curvature greater than 1. Thus, by Corollary 6.4, there exists  $\epsilon, \alpha > 0$  in  $\mathcal{B}$  such that:

$$\Delta \delta^{1+\alpha} > \epsilon(1 + \mu) - \|\text{Hess}(\delta)\| \sum_{i=1}^n \mu_i \delta_{;i} \delta_{;i}.$$

Thus, for sufficiently large  $C$ :

$$(\Delta + C\mathcal{D}_\delta)\delta^{1+\alpha} \geq \epsilon(1 + \mu).$$

This completes the proof.  $\square$

### Corollary 7.9

For  $\lambda > 0$ , define  $\Phi_\lambda = \text{Log}(a) + \lambda\delta$ . If the sectional curvature of  $M$  is bounded above by  $-1$ , then, there exists  $\lambda > 0$  and  $c > 0$  in  $\mathcal{B}(p_0)$  such that, modulo terms which vanish when  $\nabla\Phi_\lambda$  vanishes:

$$\lambda_1 \geq c \Rightarrow (\Delta + C\mathcal{D}_\delta)\Phi_\lambda > 0,$$

in the weak sense.

**Proof:** By Corollary 7.7 and Proposition 7.8, there exists  $\lambda > 0$ :

$$(\Delta + C\mathcal{D}_\delta)(\text{Log}(a) + \lambda\delta) \geq 1 - \frac{9}{8} \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2.$$

By Proposition 2.3, (iv),  $\mu_1 \lambda_1 \leq \kappa$ , and so  $\mu_1 = O(\lambda_1^{-1})$ . Thus, for all  $i \in I$ ,  $\mu_i = O(\lambda_1^{-1})$ . However, modulo terms that vanish when  $\nabla\Phi_\lambda$  vanishes, for all  $k$ :

$$\frac{1}{\lambda_1} A_{11;k} = -\lambda \delta_{;k}.$$

Thus, modulo terms that vanish when  $\nabla\Phi_\lambda$  vanishes:

$$\sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2 = \lambda^2 \sum_{i \in I} \mu_i \delta_{;i}^2 = O(\lambda_1^{-1}).$$

There thus exists  $c \geq 0$  such that, if  $\lambda_1 \geq c$ , then, modulo terms that vanish when  $\nabla\Phi_\lambda$  vanishes:

$$\frac{9}{8} \sum_{i \in I} \frac{\mu_i}{\lambda_1^2} A_{11;i}^2 < 1.$$

The result now follows.  $\square$

Alternatively, let  $R > 0$  be such that:

$$\kappa < \frac{1}{R} \mu_\infty(K).$$

### Proposition 7.10

Suppose that  $\Sigma_u \subseteq B_R(p_0)$ . Then, there exists  $c, \epsilon > 0$  in  $\mathcal{B}(p_0)$  such that:

$$\lambda_1 \geq c \Rightarrow \Delta^\Sigma \delta \geq \epsilon(1 + \sum_{i=1}^n \mu_i).$$

**Proof:** Since  $M$  has non-positive curvature, bearing in mind Proposition 2.3:

$$\begin{aligned} \text{Hess}^M(\tfrac{1}{2}\delta^2) &\geq \text{Id} \\ \Rightarrow \text{Hess}^\Sigma(\text{frac}12\delta^2) &\geq \text{Id} - d(x, x_0)\langle \mathbf{N}, \nabla d \rangle A \\ \Rightarrow \Delta \text{frac}12\delta^2 &\geq \sum_{i=1}^n \mu_i - \kappa d(x, x_0). \\ &\geq \sum_{i=1}^n \mu_i - \kappa R. \end{aligned}$$

Thus, by definition of  $R$ , there exists  $c_1, \epsilon_1 > 0$  such that, for  $\lambda_1 \geq c_1$ :

$$\Delta \tfrac{1}{2}\delta^2 \geq \epsilon_1 \sum_{i=1}^n \mu_i.$$

By Proposition 2.3, (v), there exists  $\epsilon_2 \geq 0$  such that, for  $\lambda_1 \geq c_1$ :

$$\Delta \tfrac{1}{2}\delta^2 \geq \epsilon_2(1 + \sum_{i=1}^n \mu_i).$$

This completes the proof.  $\square$

In a similar manner, we therefore obtain:

### Corollary 7.11

For  $\lambda > 0$ , define  $\Phi_\lambda = \text{Log}(a) + \tfrac{1}{2}\delta^2$ . Suppose that  $\Sigma \subseteq B_R(p_0)$ . Then there exists  $\lambda > 0$  and  $c > 0$  in  $\mathcal{B}(p_0)$  such that, modulo terms which vanish when  $\nabla \Phi_\lambda$  vanishes:

$$\lambda_1 \geq c \Rightarrow \Delta \Phi_\lambda > 0,$$

in the weak sense.

**Proof:** This is proven in a similar manner to Corollary 7.9.  $\square$

Interior bounds now follow by the maximum principal:

**Proof of Proposition 6.1:** Consider the function  $\|A\|e^{\lambda\delta} = \lambda_1 e^{\lambda\delta}$ . If this function achieves its maximum along  $\partial\Sigma$ , then the result follows since  $e^{\lambda\delta}$  is uniformly bounded above and below. Otherwise, it achieves its maximum in the interior of  $\Sigma$ , in which case, by Corollary 7.9 and the Maximum Principal, at this point:

$$\|A\| = \lambda_1 \leq c.$$

The result follows.  $\square$

**Proof of Proposition 7.2:** Consider the function  $\|A\|e^{\frac{1}{2}\lambda\delta^2}$  and proceed as in the proof of Proposition 6.1 with Corollary 7.11 used in place of Corollary 7.9.  $\square$

## 8 - Regularity of Limit Hypersurfaces.

Let  $M := M^{n+1}$  be an  $(n+1)$ -dimensional Riemannian manifold. Let  $K$  be an admissible convex curvature function. In this section, we obtain a priori estimates for the norms of the second fundamental forms of locally convex hypersurfaces of prescribed  $K$ -curvature near points where they are strictly convex. This allows us to prove smoothness and smooth convergence of limits of sequences of hypersurfaces of prescribed  $K$ -curvature near every point where the limit is strictly convex. Explicitly, let  $(\kappa_n)_{n \in \mathbb{N}}, \kappa \in C^\infty(M)$  be smooth, strictly positive functions over  $M$  such that  $(\kappa_n)_{n \in \mathbb{N}}$  converges to  $\kappa$  in the  $C_{\text{loc}}^\infty$  sense over  $M$ . Let  $(\Sigma_n)_{n \in \mathbb{N}} = (i_n, S_n)_{n \in \mathbb{N}}$  be a sequence of smooth, LSC immersed hypersurfaces in  $M$  such that, for all  $n$ ,  $\Sigma_n$  has prescribed  $K$ -curvature equal to  $\kappa_n$ . In other words, for all  $n$ :

$$K(i_n) = \kappa_n \circ i_n.$$

Suppose that there exists a  $C^{0,1}$  locally convex hypersurface  $\Sigma_0$  towards which  $(\Sigma_n)_{n \in \mathbb{N}}$  converges locally uniformly. For all  $n \in \mathbb{N}$ , let  $\mathbf{N}_n$  and  $A_n$  be the unit normal vector field and the second fundamental form respectively of  $\Sigma_n$ . Choose  $p_0 \in \Sigma_0$ , and let  $(p_n)_{n \in \mathbb{N}} \in (\Sigma_n)_{n \in \mathbb{N}}$  be a sequence converging to  $p_0$ . For all  $r > 0$  and for all  $n \in \mathbb{N} \cup \{0\}$ , let  $B_{m,r}$  be the ball of radius  $r$  (with respect to the intrinsic metric) about  $p_m$  in  $\Sigma_m$ .

We will say that  $\Sigma_0$  is **functionally strictly convex** at  $p_0$  if and only for all  $\epsilon > 0$  sufficiently small, there exists a smooth function  $f : B_\epsilon(p_0) \rightarrow \mathbb{R}$  such that:

- (i)  $f$  is strictly convex;
- (ii)  $f(p_0) > 0$ ; and
- (iii) the connected component of  $f^{-1}([0, \infty[) \cap \Sigma_0$  containing  $p_0$  is compact.

Observe that if  $M$  is affine flat (in particular, if  $M = \mathbb{H}^{n+1}$ ), then  $\Sigma_0$  is functionally strictly convex whenever it is strictly convex.

### Proposition 8.1

If  $\Sigma_0$  is functionally strictly convex at  $p_0$ , then there exists  $r > 0$  such that  $(B_{n,r}, i_n, p_n)_{n \in \mathbb{N}}$  converges to  $(B_{0,r}, i_0, p_0)$  in the  $C^\infty$  sense. In particular,  $(B_{0,r}, i_0)$  is a smooth, locally convex immersed hypersurface of prescribed  $K$ -curvature equal to  $\kappa_0$ .

**Proof:** As in Section 4, we denote by  $\mathcal{B}$  the family of constants which depend continuously on the data, being in this case  $M$ ,  $K$ ,  $\kappa$  ( $\Sigma_0, p_0$ ) and the  $C^1$  jets of  $(\Sigma_m, p_m)_{m \in \mathbb{N}}$ , and for any quantity,  $X$ , we denote by  $O(X)$  any term which is bounded in magnitude by  $K|X|$  for some  $K$  in  $\mathcal{B}$ . By hypothesis, we may assume that there exists  $\epsilon > 0$  and, for all  $n \in \mathbb{N} \cup \{0\}$  a smooth function  $f_n : B_\epsilon(p_n) \rightarrow \mathbb{R}$  such that:

- (i)  $(f_n)_{n \in \mathbb{N}}$  converges to  $f_0$  in the  $C^\infty$  sense;
- (ii) for all  $n$ ,  $\text{Hess}(f_n) \geq \epsilon \text{Id}$ ;
- (iii) for all  $n$ ,  $f_n(p_n) = 2h$ ; and

(iv) for all  $n$ , the connected component of  $p_n$  in  $\Sigma_n \cap f^{-1}([0, \infty[)$  is compact: we denote this connected component by  $\Sigma_{n,0}$ .

We may assume that, for all  $n$ ,  $f_n \leq 1$  over  $\Sigma_{n,0}$ . Moreover, without loss of generality, we may assume that, for all  $n$ ,  $\kappa_n > \epsilon$ , and that there exists a smooth unit length vector field  $X$  defined in a neighbourhood of  $p_0$  such that, for all  $n$ , throughout  $\Sigma_{n,0}$ ,  $\langle X, \mathbf{N}_n \rangle \geq 2\epsilon$ . We now follow the approach of Sheng, Urbas and Wang, [21], which itself is a development of the work, [18], of Pogorelov (see also [22]).

Choose  $\alpha \geq 1$ . For all  $m$ , we define the function  $\Phi_m$  by:

$$\Phi_m = \text{Log}(\|A_m\|) - \text{Log}(\langle X, \mathbf{N}_n \rangle - \epsilon) + \alpha \text{Log}(f_m),$$

where  $\|A_m\|$  is the operator norm of  $A_m$ , which is equal to its highest eigenvalue. We aim to obtain a priori upper bounds for  $\Phi_m$  for some  $\alpha$ . We trivially obtain a priori bounds whenever  $\|A_m\| \leq 1$ , and we thus consider the region where  $\|A_m\| \geq 1$ . Choose  $n \in \mathbb{N}$  and  $p \in \Sigma_{m,0}$ . Let  $\lambda_1 \geq \dots \geq \lambda_n$  be the eigenvalues of  $A_m$  at  $P$ . In particular,  $\lambda_1 = \|A_m\|$ . Let  $e_1, \dots, e_n$  be the corresponding orthonormal basis of eigenvectors. In the sequel, we will suppress  $m$ .

Let the subscript  $;$  denote covariant differentiation with respect to the Levi-Civita covariant derivative of  $\Sigma$ . Thus, for example:

$$A_{ij;k} = (\nabla_{e_k}^\Sigma A)(e_i, e_j).$$

By Proposition 2.3, (ii), we may assume that  $DK_A$  is diagonal with respect to  $e_1, \dots, e_n$ . Let  $\mu_1 \leq \dots \leq \mu_n$  be the eigenvalues of  $DK_A$  at  $p$ . We consider the Laplacian,  $\Delta$ , defined on functions over  $\Sigma$  by:

$$\Delta f = DK_A(\text{Hess}^\Sigma(f)) = \sum_{i=1}^n \mu_i f_{;ii}.$$

We aim to use the Maximum Principal in conjunction with  $\Delta$ . Thus, in the sequel, we will only be interested in the orders of magnitude of potentially negative terms.

We consider the three terms in  $\Phi$  separately. Choose  $\delta \in ]0, 1/4[$  such that  $\delta < \epsilon^2$ . By Proposition 2.3, (iv):

$$\sum_{i=1}^n \mu_i \lambda_i = \kappa \geq \epsilon.$$

Thus, as in the proof of Proposition 7.6:

$$\begin{aligned} \Delta \text{Log}(\lambda_1) &= \frac{1}{\lambda_1} \kappa_{;11} - \frac{1}{\lambda_1} (D^2 K)^{ij,mn} A_{ij;1} A_{mn;1} - \sum_{i=1}^n \frac{\mu_i}{\lambda_1^2} A_{11;i} A_{11;i} \\ &\quad + \sum_{i,j=1}^n \frac{\mu_i}{\lambda_1} (R_{1ii}^\Sigma A_{p1} + R_{1i1}^\Sigma A_{pi}) + O(1) + O(\mu). \\ &\geq -\frac{1}{\lambda_1} (D^2 K)^{ij,mn} A_{ij;1} A_{mn;1} - \frac{1}{\lambda_1^2} (1 + \delta) \sum_{i=1}^n \mu_i A_{i1;1}^2 \\ &\quad + \epsilon \lambda_1 - \sum_{i=1}^n \mu_i \lambda_i^2 + O(1) + O(\mu). \end{aligned}$$



Next, let  $d$  be the signed distance in  $M$  to  $\Sigma$ . Thus, along  $\Sigma$ ,  $\nabla d = \mathbf{N}$ . Then:

$$\text{Hess}^M(\langle X, \nabla d \rangle)_{pq} = X^i{}_{;pq} d_{;i} + X^i{}_{;p} d_{;iq} + X^i{}_{;q} d_{;ip} + X^i d_{;ipq}.$$

However:

$$d_{;ipq} = d_{;piq} = d_{;pqi} + R^M{}_{iqp}{}^m d_{;m}.$$

By Proposition 6.6, Proposition 6.5, (i) and Lemma 7.3, along  $\Sigma$ , for  $1 \leq j \leq n$ :

$$\begin{aligned} \sum_{i=1}^n \mu_i d_{;jii} &= \sum_{i=1}^n \mu_i (d_{;iij} + R_{ij\nu i}) \\ &= \sum_{i=1}^n \mu_i (A_{;iij} + R_{ij\nu i}) \\ &= \kappa_{;j} + O(\mu) \\ &= O(1) + O(\mu). \end{aligned}$$

Likewise, by Proposition 6.6 and Lemma 7.3:

$$\sum_{i=1}^n \mu_i X^{;\nu} d_{;\nu ii} = - \sum_{i=1}^n \mu_i X^{;\nu} A_{ii}^2 = - \langle X, \mathbf{N} \rangle \sum_{i=1}^n \mu_i \lambda_i^2.$$

Thus, bearing in mind Proposition 2.3, (iv):

$$\sum_{i=1}^n \mu_i \text{Hess}^M(\langle X, \mathbf{N} \rangle)_{ii} = - \langle X, \mathbf{N} \rangle \sum_{i=1}^n \mu_i \lambda_i^2 + O(1) + O(\mu).$$

Finally:

$$\text{Hess}^\Sigma(\langle X, \mathbf{N} \rangle) = \text{Hess}^M(\langle X, \mathbf{N} \rangle) - \langle \nabla \langle X, \mathbf{N} \rangle, \mathbf{N} \rangle A.$$

However, by Proposition 6.6, (i):

$$\langle \nabla \langle X, \mathbf{N} \rangle, \mathbf{N} \rangle = X^\nu{}_{;\nu} + X^i f_{;i\nu} = X^\nu{}_{;\nu} = O(1).$$

Thus, by Proposition 2.3, (iv):

$$\Delta \langle X, \mathbf{N} \rangle = - \langle X, \mathbf{N} \rangle \sum_{i=1}^n \mu_i \lambda_i^2 + O(1) + O(\mu).$$

In like manner:

$$(\nabla_i \langle X, \mathbf{N} \rangle)^2 = O(1) + O(1) \lambda_i + (X^i)^2 \lambda_i^2.$$

Consequently, by Proposition 2.3, (iv), and bearing in mind that  $2\epsilon \leq \langle X, \mathbf{N} \rangle \leq 1$ :

$$\begin{aligned} -\Delta \text{Log}(\langle X, \mathbf{N} \rangle - \epsilon) &= \frac{\langle X, \mathbf{N} \rangle}{\langle X, \mathbf{N} \rangle - \epsilon} \sum_{i=1}^n \mu_i \lambda_i^2 + \frac{1}{(\langle X, \mathbf{N} \rangle - \epsilon)^2} \sum_{i=1}^n (X^i)^2 \mu_i \lambda_i^2 + O(1) + O(\mu) \\ &\geq \sum_{i=1}^n \mu_i \lambda_i^2 + \frac{1+\epsilon^2}{(\langle X, \mathbf{N} \rangle - \epsilon)^2} \sum_{i=1}^n (X^i)^2 \mu_i \lambda_i^2 + O(1) + O(\mu). \end{aligned}$$

Finally:

$$\Delta \alpha \text{Log}(f) = \sum_{i=1}^n \mu_i \alpha \frac{1}{f} \text{Hess}^M(f)_{ii} - \sum_{i=1}^n \alpha \mu_i \frac{f_{;i}^2}{f^2} - \frac{\alpha f_\nu}{f} \sum_{i=1}^n \mu_i \lambda_i.$$

Thus, by Proposition 2.3, (iv), and uniform convexity of  $f$ , and bearing in mind that  $f \leq 1$ :

$$\Delta \alpha \text{Log}(f) \geq \alpha \epsilon \mu - \sum \alpha \mu_i f_{;i}^2 f^{-2} + O(\alpha f^{-1}).$$

Combining these terms, we obtain:

$$\Delta \Phi \geq A_1 + B_1 + C_1,$$

where:

$$\begin{aligned} A_1 &= \epsilon \lambda_1 + O(\alpha f^{-1}) \\ B_1 &= \alpha \epsilon \mu + O(1) + O(\mu) + \frac{(1+\epsilon^2)}{(\langle X, \mathbf{N} \rangle - \epsilon)^2} \sum_{i=1}^n (X^i)^2 \mu_i \lambda_i^2 \\ C_1 &= -\frac{1}{\lambda_1} (D^2 K)^{ij;mn} A_{ij;1} A_{mn;1} - \frac{(1+\delta)}{\lambda_1^2} \sum_{i=1}^n \mu_i A_{i1;1}^2 - \sum \alpha \mu_i f_{;i}^2 f^{-2}. \end{aligned}$$

The potentially bad terms are contained in term  $C_1$ . We now aim to eliminate them. Define  $I, J \subseteq \{1, \dots, n\}$  by:

$$I = \{i \text{ s.t. } \mu_i \leq 4\mu_1\}, \quad J = \{i \text{ s.t. } \mu_i > 4\mu_1\}.$$

Bearing in mind Lemma 7.3, (i), differentiating  $\Phi$  yields:

$$\begin{aligned} \Phi_{;i} &= \frac{1}{\lambda_1} A_{i1;1} + \frac{1}{\lambda_1} R_{i1\nu 1} - \frac{1}{(\langle X, \mathbf{N} \rangle - \epsilon)} X^i \lambda_i \\ &\quad - \frac{1}{(\langle X, \mathbf{N} \rangle - \epsilon)} (\nabla^M X)^\nu_i + \frac{\alpha}{f} f_{;i}. \end{aligned}$$

Thus, bearing in mind that  $\lambda_1 \geq 1$ , modulo terms that vanish when  $\nabla \Phi$  vanishes, for all  $\eta > 0$ :

$$\frac{1}{\lambda_1^2} A_{i1;1}^2 \leq \frac{(1+\eta)}{(\langle X, \mathbf{N} \rangle - \epsilon)^2} (X^i)^2 \lambda_i^2 + O(1) + O\left(\frac{\alpha^2}{f^2}\right).$$

We choose  $\eta > 0$  such that:

$$(1+\delta)(1+\eta) < (1+\epsilon^2).$$

Since  $\mu_1 \lambda_1 \leq \kappa$ , and since, for all  $i \in I$ ,  $\mu_i \leq 4\mu_1$ , for all  $i \in I$ , bearing in mind that  $\lambda_1 \geq 1$ :

$$\mu_i = O(\lambda_1^{-1}) = O(1).$$

Thus:

$$\Delta \Phi \geq A_2 + B_2 + C_2,$$

where:

$$\begin{aligned} A_2 &= \epsilon \lambda_1 + O(\alpha f^{-1}) + O(\alpha^2 f^{-2}) \\ B_2 &= \alpha \epsilon \mu + O(1) + O(\mu) + \frac{(1+\epsilon^2)}{(\langle X, \mathbf{N} \rangle - \epsilon)^2} \sum_{i \in J} (X^i)^2 \mu_i \lambda_i^2 \\ C_2 &= -\frac{1}{\lambda_1} (D^2 K)^{ij;mn} A_{ij;1} A_{mn;1} - \frac{(1+\delta)}{\lambda_1^2} \sum_{i \in J} \mu_i A_{i1;1}^2 - \sum_{i \in J} \alpha \mu_i f_{;i}^2 f^{-2}. \end{aligned}$$

Likewise, bearing in mind that  $\lambda_1 \geq 1$ , modulo terms which vanish when  $\nabla \Phi$  vanishes:

$$\frac{\alpha f_{;i}^2}{f^2} \leq \frac{4}{\alpha \lambda_1^2} A_{i1;1}^2 + \frac{4}{\alpha (\langle X, \mathbf{N} \rangle - \epsilon)^2} (X^i)^2 \lambda_i^2 + O\left(\frac{1}{\alpha}\right).$$

Thus, for  $\alpha \geq 4\delta^{-1} \geq 4$ :

$$\Delta\Phi \geq A_3 + B_3 + C_3,$$

where:

$$\begin{aligned} A_3 &= \epsilon\lambda_1 + O(\alpha^{-1}) + O(\alpha f^{-1}) + O(\alpha^2 f^{-2}) \\ B_3 &= \alpha\epsilon\mu + O(1) + O(\mu) \\ C_3 &= -\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} - \frac{(1+2\delta)}{\lambda_1^2} \sum_{i \in J} \mu_1 A_{i1;1}^2. \end{aligned}$$

By Proposition 2.3, (vi):

$$-\frac{1}{\lambda_1}(D^2K)^{ij,mn}A_{ij;1}A_{mn;1} \geq \frac{2}{\lambda_1} \sum_{j \in J} \frac{\mu_j - \mu_1}{\lambda_1 - \lambda_j} A_{j1;1}^2.$$

However, for  $j \in J$ ,  $\mu_j \geq 4\mu_1$ , and so:

$$\frac{2(\mu_j - \mu_1)}{\lambda_1(\lambda_1 - \lambda_j)} - \frac{(1+2\delta)\mu_j}{\lambda_1^2} \geq \frac{(1-2\delta)\mu_j\lambda_1 - 2\mu_1\lambda_1}{\lambda_1^2(\lambda_1 - \lambda_j)} \geq 0.$$

Thus:

$$C_3 \geq 0.$$

By Proposition 2.3, (v), we may assume that  $\mu \geq \epsilon$ , and thus, for sufficiently large  $\alpha$ ,  $B_3 \geq 0$ , and so:

$$\begin{aligned} \Delta\Phi &\geq \epsilon\lambda_1 + O(1) + O(f^{-1}) + O(f^{-2}) \\ &= (f^{2\alpha}\lambda_1)^{-1}(\epsilon(f^\alpha\lambda_1)^2 + O(f^\alpha\lambda_1) + O(1)). \end{aligned}$$

There therefore exists  $K_1 > 0$  in  $\mathcal{B}$  such that, if  $(f^\alpha\|A\|) \geq K$ , then the right hand side is positive. However, for all  $m \in \mathbb{N}$ ,  $\Phi_m = -\infty$  along  $\partial\Sigma_{m,0}$ . There thus exists a point  $P \in \Sigma_{m,0}$  where  $\Phi_m$  is maximised. By the Maximum Principal, at this point, either  $\|A\| \leq 1$  or  $f^\alpha\|A\| \leq K_1$ . Taking exponentials, there therefore exists  $K_2 > 0$  in  $\mathcal{B}$  such that, for all  $m \in \mathbb{N}$ , throughout  $\Sigma_{m,0}$ :

$$f_m^\alpha(\langle X, \mathbf{N}_m \rangle - \epsilon)^{-1}\|A_m\| \leq K_2.$$

Since  $\langle X, \mathbf{N} \rangle - \epsilon \leq 1$ , this yields a-priori bounds for  $\|A_m\|$  over the intersection of  $\Sigma_{m,0}$  with  $f_m \geq h$ . Using, for example, an adaptation of the proof of Theorem 1.2 of [23] in conjunction with the Bernstein Theorem [6], [14] & [18] of Calabi, Jörgens, Pogorelov, we obtain a-priori  $C^k$  bounds for  $\Sigma_{m,0}$  over the region  $f_m \geq 3h$  for all  $k$ . The result now follows by the Arzela-Ascoli Theorem.  $\square$

## 9 - Existence.

We now prove the main results of this paper.

**Proof of Theorem 1.1:** Suppose that there exists no smooth, LSC immersed hypersurface,  $\Sigma$ , distinct from  $(\Sigma_l, \partial\Sigma_l)$ , satisfying Conditions (i), (ii) and (a). We extend  $\kappa_0$  to

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a smooth family of strictly positive functions,  $(\kappa_t)_{t \in [0,1]}$  such that  $\kappa_1 = \kappa$ , and, for all  $t \in ]0, 1[$ :

- (i)  $K(\Sigma_u) > \kappa_t$ ; and
- (ii)  $K(\Sigma_l) < \kappa_t$ .

Consider the family,  $\mathcal{F}$ , of smooth, LSC, immersed hypersurfaces  $\Sigma := (\Sigma, \partial\Sigma)$  such that:

$$\Sigma_u > \Sigma > \Sigma_l.$$

We give  $\mathcal{F}$  the topology of  $C^\infty$  convergence. For  $\epsilon > 0$ , define  $X \subseteq \mathcal{F} \times [\epsilon, 1]$  to be the set of all pairs  $(\Sigma, t)$  such that  $\Sigma_l < \Sigma < \Sigma_u$  and:

$$K(\Sigma) = \kappa_t.$$

Note that we cannot take  $\epsilon = 0$  since  $\Sigma_l$  does not strictly bound itself. We claim that  $X$  is compact. Indeed, let  $(\Sigma_n, t_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . We may suppose that  $(t_n)_{n \in \mathbb{N}}$  converges to  $t_\infty$ . Proposition 3.8 provides uniform  $C^{0,1}$  bounds for  $(\Sigma_n)_{n \in \mathbb{N}}$ ; Proposition 6.1 then provides uniform  $C^2$  bounds for  $(\Sigma_n)_{n \in \mathbb{N}}$  along the boundary; Proposition 7.1 then provides global uniform  $C^2$  bounds for  $(\Sigma_n)_{n \in \mathbb{N}}$ ; global  $C^{2,\alpha}$  bounds for  $(\Sigma_n)_{n \in \mathbb{N}}$  follow from Theorem 1 of [3] and global  $C^k$  bounds for all  $k$  follow from the Schauder Estimates (c.f. [29]). It follows by the Arzela-Ascoli Theorem that there exists a smooth, LSC immersed hypersurface  $\Sigma_\infty$  towards which  $(\Sigma_n)_{n \in \mathbb{N}}$  converges in the  $C^\infty$  sense. Trivially:

$$K(\Sigma_\infty) = \kappa_{t_\infty}.$$

Bearing in mind that  $\mathcal{F}$  is not closed, it remains to show that  $\Sigma_\infty \in \mathcal{F}$ . Trivially:

$$\Sigma_u \geq \Sigma_\infty \geq \Sigma_l.$$

We claim that  $\Sigma_\infty$  is nowhere tangent to  $\Sigma_u$ . Indeed, suppose the contrary, then, by the Geometric Maximum Principal, at the point of tangency:

$$K(\Sigma_u) \leq K(\Sigma_\infty) = \kappa_t < K(\Sigma_u),$$

which is absurd, and the assertion follows. Likewise,  $\Sigma_\infty$  is nowhere tangent to  $\Sigma_l$ , and so:

$$\Sigma_u > \Sigma_\infty > \Sigma_l.$$

We conclude that  $\Sigma_\infty \in \mathcal{F}$  and  $X$  is thus compact.

Let  $DK$  be the linearisation of the  $K$ -curvature operator. Consider  $(\Sigma, \partial\Sigma) = (i, (S, \partial S)) \in \mathcal{F}$  and let  $\mathbf{N}$  be the outward pointing unit vector field over  $\Sigma$ . or all  $t$ , we define the operator,  $\mathcal{L}_t$ , on functions over  $S$  by:

$$\mathcal{L}_t f := \mathcal{L}_{\kappa_t} f = DKf - \langle \nabla \kappa_t, \mathbf{N} \rangle f,$$

By hypothesis,  $\mathcal{L}_0$  is invertible. We claim that, for all  $\epsilon > 0$  sufficiently small, there exists at most one hypersurface  $\Sigma_\epsilon \in \mathcal{F}$  such that  $(\Sigma_\epsilon, \epsilon) \in X$ . Indeed, by hypothesis,  $\Sigma_l$  is the unique LSC immersed hypersurface lying between  $\Sigma_l$  and  $\Sigma_u$  such that:

$$K(\Sigma_l) = \kappa_0.$$

Thus, if there exist two solutions for a sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  converging to 0, then, by compactness again, both these sequences of solutions converge to  $\Sigma_0$ . However, since  $\mathcal{L}_0$  is invertible, it follows by the Implicit Function Theorem for Banach spaces that these two sequences of immersed hypersurfaces coincide for sufficiently large  $n$ , which is absurd, and the assertion follows.

Likewise, for all  $\epsilon$  sufficiently small, if  $(\Sigma_\epsilon, \epsilon) \in X$ , then  $\mathcal{L}_\epsilon$  is invertible over  $\Sigma_\epsilon$ . Indeed, assume the contrary, then there exists a sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  converging to 0 with solutions for which  $\mathcal{L}_{\epsilon_n}$  is not invertible. However, by compactness again, this sequence converges to  $\Sigma_0$  and  $(\mathcal{L}_{\epsilon_n})_{n \in \mathbb{N}}$  converges to  $\mathcal{L}_0$ . However,  $\mathcal{L}_0$  is invertible over  $\Sigma_0$ , which is absurd, and the assertion follows.

Finally, it follows by the Implicit Function Theorem for Banach Spaces and the positivity of  $\mathcal{L}_0$ , that for all sufficiently small  $\epsilon > 0$ , there exists an LSC immersed hypersurface  $\Sigma_\epsilon$  such that  $(\Sigma_\epsilon, \epsilon) \in X$ .

Consequently, for all sufficiently small  $\epsilon$ , all hypersurfaces  $\Sigma_\epsilon$  such that  $(\Sigma_\epsilon, \epsilon) \in X$  are non-degenerate, and the number of such solutions counted modulo 2 is equal to 1. It now follows by mod 2 degree theory as developed in [22] and [26] that there exists a smooth, LSC immersed hypersurface  $\Sigma \in \mathcal{F}$  such that:

$$K(\Sigma) = \kappa_1 = \kappa.$$

This completes the proof.  $\square$

**Proof of Theorem 1.2:** We first claim that  $\hat{\Sigma}$  is isotopic through smooth, LSC, immersed hypersurfaces to an immersed submanifold in a geodesic sphere. Indeed, let  $\mathbf{N}_{\hat{\Sigma}}$  be the outward pointing unit normal vector field over  $\hat{\Sigma}$  and define  $I : \hat{\Sigma} \times [0, \infty[ \rightarrow M$  by:

$$I(p, t) = \text{Exp}(t\mathbf{N}_{\hat{\Sigma}}(p)).$$

Let  $d : \hat{\Sigma} \times [0, \infty[ \rightarrow \mathbb{R}$  be the distance in  $\hat{\Sigma} \times [0, \infty[$  to  $\hat{\Sigma} \times \{0\}$ .  $d$  is a convex function. For all  $t \in [0, \infty[$ , let  $\hat{\Sigma}_t = d^{-1}(\{t\})$  be the level set of  $d$  at height  $t$ . Choose  $p \in M$  and let  $d_p$  be the distance to  $p$  in  $M$ .  $d_p$  is also a convex function and we identify it with  $d_p \circ I$ .

Observe that  $\partial(\hat{\Sigma} \times [0, \infty[)$  consists of 2 components, being  $\hat{\Sigma}$  and  $(\partial\hat{\Sigma}) \times [0, \infty[$ . Choose  $R \geq 0$  such that, for all  $q \in \hat{\Sigma}_0$ :

$$d_p(q) < R.$$

Observe that, since  $M$  is non-positively curved, as  $d(q)$  tends to  $+\infty$ , the angle between  $\nabla d$  and  $\nabla d_p$  at  $q$  tends to 0. Thus, increasing  $R$  if necessary, we may assume that, for  $d_p(q) \geq R$ :

$$\langle \nabla d, \nabla d_p \rangle(q) > 0.$$

For  $s \in [0, 1]$  define  $d_s$  and  $\Sigma_s$  by:

$$d_s = s d_p + (1 - s) d, \quad \Sigma_s = d_s^{-1}(\{R\}).$$

For all  $s$ , and for all  $q$  such that  $d_p(q) \geq R$ :

$$\langle \nabla d, \nabla d_s \rangle(q) > 0.$$

Thus, for all  $s \in [0, 1]$ ,  $\Sigma_s \cap \hat{\Sigma}_0 = \emptyset$  and  $\Sigma_s$  is transverse to  $\partial \hat{\Sigma} \times [0, \infty[$ . Moreover, for all  $s$ ,  $d_s$  is convex, and so  $(\Sigma_s)_{s \in [0, 1]}$  defines an isotopy through smooth, compact, LSC immersed hypersurfaces from  $\hat{\Sigma}_R = \Sigma_0$  to  $\Sigma_1 \subseteq \partial B_R(p)$ . Since  $\hat{\Sigma}_0$  is isotopic to  $\hat{\Sigma}_R$ , the assertion follows.

Using the a-priori estimates developed in this paper (Propositions 6.1 and 7.1), we now proceed as in the proof of Theorem 1.1 of [26]. The only supplementary result required is a proof that if  $\Sigma_0$  is a uniform limit of smooth, LSC, immersed hypersurfaces of prescribed  $K$ -curvature, then its normal at any boundary point lies away from the “minimal convex direction” of  $\Gamma := \partial \Sigma$  (what we call the “convexity orientation” in Section 4 of [26]). If  $K$  is of finite type, then we easily adapt Proposition 4.1 to yield this result (it is even unnecessary to perturb  $\Phi_0$ ). If  $K$  is of determinant type, then we likewise easily adapt Proposition 9.1 of [26] to yield this result. This completes the proof.  $\square$

**Proof of Theorem 1.3:** Let  $\kappa : M \rightarrow ]0, 1[$  be a smooth function. Let  $\Sigma = (i, (S, \partial S))$  be an isometrically immersed LSC hypersurface such that  $K(\Sigma) = \kappa$ . Let  $\mathbf{N}$  be the outward pointing unit normal over  $\Sigma$ . Let  $\mathcal{L}$  be the linearisation of the  $K$ -curvature operator over  $\Sigma$ . By Proposition 3.1.1 of [15], for all  $f \in C^\infty(S)$ :

$$\mathcal{L}f = (DK_A(W) - DK_A(A^2))f - DK_A(\text{Hess}(f)),$$

where  $W : TS \rightarrow TS$  is given by:

$$W \cdot X = R_{\mathbf{N}X} \mathbf{N}.$$

Since the sectional curvature of  $M$  is bounded above by  $-1$ , for all  $X \in TS$ :

$$\begin{aligned} \langle W \cdot X, X \rangle &= \langle R_{\mathbf{N}X} \mathbf{N}, X \rangle \\ &\geq \|X\|^2 \\ \Rightarrow W &\geq \text{Id} \\ \Rightarrow DK_A(W) &\geq DK_A(\text{Id}). \end{aligned}$$

It thus follows from the hypotheses on  $K$  that:

$$DK_A(W) - DK_A(A^2) > 0.$$

Thus, for all  $f \in C^\infty(S)$ :

$$\langle \mathcal{L}f, f \rangle \geq 0,$$

with equality if and only if  $f = 0$ .

We may conclude in two different ways. First, we may interpret this in terms of the degree theory of [19], in which case we see that the contribution of any solution to the degree of  $\kappa$  is equal to +1, and that there is therefore only one solution. Alternatively, we reason more directly, as in the proof of Lemma 3.0.2 of [15], to reach the same conclusion. This completes the proof.  $\square$

**Proof of Theorem 1.4:** We claim that if  $\Sigma_u > \Sigma > \Sigma_l$ , then  $\Sigma$  is contained in  $B_R(p_0)$ . Indeed, let  $(I, N)$  be the convex cobordism from  $\Sigma_l$  to  $\Sigma_u$ . By definition,  $N$  is foliated by geodesics normal to  $\Sigma_l$  and terminating in  $\Sigma_u$ . Let  $\gamma : [0, 1] \rightarrow N$  be such a geodesic. Then:

$$(I \circ \gamma)(0) \in \Sigma_l \subseteq B_R(p_0), \quad (I \circ \gamma)(1) \in \Sigma_u \subseteq B_R(p_0).$$

However, since  $M$  is non-positively curved,  $B_R(p_0)$  is convex, and so:

$$(I \circ \gamma) \subseteq B_R(p_0).$$

Since  $\gamma$  is arbitrary:

$$(I(N)) \subseteq B_R(p_0).$$

Now let  $N_l := (I_l, N_l)$  and  $N_u := (I_u, N_u)$  be the convex cobordisms from  $\Sigma_l$  to  $\Sigma$  and from  $\Sigma$  to  $\Sigma_u$  respectively. By Proposition 3.6,  $N_l \cup N_u$  is the convex cobordism from  $\Sigma_l$  to  $\Sigma_u$ , and thus, by uniqueness,  $N = N_l \cup N_u$ . In particular,  $\Sigma \subseteq N$ , and so:

$$\Sigma \subseteq B_R(p_0).$$

The assertion follows, and the proof is now identical to the proof of Theorem 1.1 with Proposition 7.1 replaced by Proposition 7.2.  $\square$

**Proof of Theorem 1.5:** Let  $\Sigma := (i, (S, \partial S))$ . We claim that if  $\Sigma_u > \Sigma$ , then  $\Sigma \subseteq B_R(p_0)$ . Indeed, suppose the contrary. Define  $R_0$  by:

$$R_0 = \sup \{r \text{ s.t. } \Sigma \subseteq B_r(p_0)\}.$$

By assumption,  $R_0 > R$ , and so  $\partial B_{R_0}(p_0) \cap \partial \Sigma = \emptyset$ . It follows that  $\Sigma$  is an interior tangent to  $B_{R_0}(p_0)$  at some point. Let  $\gamma : [0, \infty[ \rightarrow M$  be the half geodesic leaving  $B_{R_0}(p_0)$  and normal to  $\partial B_{R_0}(p_0)$  at this point. Since  $N$  is a Hadamard manifold,  $B_{R_0}(p_0)$  is convex, and so  $\gamma$  never meets  $B_{R_0}(p_0)$  again. Since  $N$  is compact, by Proposition 3.5,  $\gamma$  leaves  $N$  through  $\Sigma_u$  and thus, in particular, intersects  $\Sigma_u$  at some point, which is absurd, since  $\Sigma_u \subseteq B_{R_0}(p_0)$ . The assertion follows, and the proof is now identical to the proof of Theorem 1.2 with Proposition 7.1 replaced by Proposition 7.2.  $\square$

When  $M$  is affine flat, using approximation, we refine Theorem 1.4 to the following result:

### Lemma 9.1

Let  $M$  be an affine flat manifold of non-positive sectional curvature. Let  $X \subseteq M$  be a compact subset. There exists  $r > 0$  which only depends on  $K$ ,  $\kappa$  and  $X$  with the following property:

Choose  $p_0 \in X$ . Let  $H$  be a totally geodesic hyperplane in  $M$  passing through  $p_0$ . If  $\Omega$  is an open subset of  $H$  with smooth boundary, and if  $\Sigma_u$  is a  $C^{0,1}$ , LSC, immersed hypersurface in  $M$  such that:

- (i)  $\Sigma_u$  is a piecewise smooth hypersurface consisting of a finite number of components which intersect transversally;
- (ii)  $K(\Sigma_u) > \kappa$  in the viscosity sense;
- (iii)  $\Sigma_u > \Omega$ ; and
- (iv)  $\Sigma_u$  and  $\Omega$  are contained in  $B_r(p_0)$ ,

then there exists a smooth, LSC immersed hypersurface  $\Sigma$  such that:

- (i)  $\Omega < \Sigma < \Sigma_u$ ; and
- (ii)  $K(\Sigma) = \kappa \circ i$ .

**Proof:** We first deform the metric on  $M$  slightly so that it has strictly negative sectional curvature. Since  $M$  is affine flat, for  $r$  sufficiently small,  $B_r(p_0)$  is foliated by totally geodesic hyperplanes parallel to  $H$ . We deform these hyperplanes to obtain a foliation  $(H_t)_{t \in ]-\epsilon, \epsilon[}$  of smooth, LSC hypersurfaces. Define  $\kappa_0 \in C^\infty(B_r(p_0))$  such that, for all  $p \in B_r(p_0)$ ,  $\kappa_0(p)$  is the  $K$ -curvature of the leaf of this foliation passing through  $p$ . For a sufficiently small deformation,  $\kappa_0 < \kappa$ . Moreover, by calculating  $DK$  as in Lemma 7.2 of [25], we may assume that  $\mathcal{L}_{\kappa_0}$  is non-degenerate and stable over every open subset of  $H_0$  with smooth boundary.

By Lemma 2.13 of [24], we may approximate  $\Sigma_u$  by a sequence  $(\Sigma_{u,n})_{n \in \mathbb{N}}$  of smooth, LSC immersed hypersurfaces such that, for all  $n$ :

$$K(\Sigma_{u,n}) > \kappa.$$

For all  $n$ , let  $\Omega_n \subseteq H_0$  be an open subset of  $H_0$  with smooth boundary such that  $\partial\Omega_n = \partial\hat{\Sigma}_n$ . Suppose that  $\Sigma$  is a LSC smooth hypersurface which is a graph over  $\Omega_n$  such that  $\Omega_n \leq \Sigma < \Sigma_{u,n}$  and  $K(\Sigma) = \kappa_0$ . By the Geometric Maximum Principal, there exists  $t \geq 0$  such that  $\Sigma$  is in fact an open subset of  $H_t$ . Since  $\partial\Sigma$  is contained in  $H_0$ , it follows that so is  $\Sigma$ . Thus,  $\Omega_n$  is the only LSC smooth hypersurface which is a graph over  $\Omega_n$  lying below  $\hat{\Sigma}_n$  of prescribed  $K$ -curvature equal to  $\kappa_0$ . We thus exclude possibility (a) of Theorem 1.1, and thus, for all  $n$ , there exists a smooth, LSC immersed hypersurface  $\Sigma_n$  which is a graph over  $\Omega$  such that:

- (i)  $\Sigma_n < \Sigma_{u,n}$ ; and
- (ii)  $K(\Sigma_n) = \kappa$ .

Letting  $n$  tend to infinity, letting  $\delta$  tend to 0 and using a diagonal argument, we show that there exists a locally convex immersed hypersurface  $\Sigma$  which is a graph over  $\Omega$  such that:

- (i)  $\Sigma \leq \Sigma_u$ ; and
- (ii)  $K(\Sigma) = \kappa$  in the viscosity sense.



By Proposition 8.1,  $\Sigma$  is smooth near any point where it is strictly convex. Moreover, the set of points where  $\Sigma$  is not strictly convex is closed. As in the proof of Lemma 6.5 of [25], we show that this set is stratified by totally geodesic strata which are the convex hulls of subsets of the boundary. Thus, by convexity, if  $\Sigma$  is not strictly convex at a single point, then it coincides with  $\Omega$ , contradicting the fact that it is a viscosity solution over its interior. We deduce that  $\Sigma$  is strictly convex everywhere over its interior, and is thus smooth. This completes the proof.  $\square$

This yields Theorem 1.7

**Proof of Theorem 1.7:** The Perron method as used in [8] and [29] relies on three results:

- (i) a compactness result for families of LSC immersions with prescribed smooth boundary. This is the “Main Lemma” of [29]. Although Trudinger and Wang only prove this result for hypersurfaces immersed in  $\mathbb{R}^{n+1}$  it immediately extends to any Hadamard manifold for which there exists a totally geodesic hypersurface passing through any point and normal to any unit vector at that point. This condition is trivially satisfied when  $M$  is affine flat;
- (ii) the existence result, [7], of Guan, which is replaced in the current context by Lemma 9.1; and
- (iii) the regularity result, [21], of Sheng, Urbas and Wang, which describes when sequences of LSC hypersurfaces of prescribed curvature converge smoothly to a smooth limit, and which is replaced in the current context by Proposition 8.1.

We thus have all the ingredients we require to apply the Perron method, and the result now follows as in [8] and [29].  $\square$

## 10 - On Work of Guan, Spruck and Szapiel.

In this section, we prove Theorem 1.8:

**Proof of Theorem 1.8:** This follows immediately from Proposition 10.1 and Corollary 10.3.  $\square$

### Proposition 10.1

With the hypotheses of Theorem 1.8, there exists a  $C^1$  LSC, embedded, hypersurface  $\Sigma$  in  $\mathbb{H}^{n+1}$  such that:

- (i)  $\Sigma$  is  $C^\infty$  away from the boundary;
- (ii)  $\partial\Sigma = \Gamma$ ; and
- (iii)  $K(\Sigma) = k$ .

**Proof:** We identify  $\mathbb{H}^{n+1}$  with the upper half space in  $\mathbb{R}^{n+1}$ . For all  $\epsilon > 0$ , define the horosphere  $H_\epsilon = \mathbb{R}^n \times \{\epsilon\}$  and define  $\Gamma_\epsilon \subseteq H_\epsilon$  by:

$$\Gamma_\epsilon = \{(x, \epsilon) \text{ s.t. } x \in \Gamma\}.$$

Let  $(\epsilon_n)_{n \in \mathbb{N}} > 0$  be a sequence of positive numbers converging to 0. By Theorem 1.3 of [10], we may assume that, for all  $n$ , there exists a smooth, LSC, embedded hypersurface,  $\Sigma_n$ , which is a graph of a smooth, function over  $\Omega$  such that, for all  $n$ :

- (i)  $\partial\Sigma_n = \Gamma$ ; and
- (ii)  $K(\Sigma_n) = k$ .

For all  $n$ ,  $\Sigma_n$  bounds a convex set. Thus, taking limits, we may assume that there exists a  $C^{0,1}$  convex, embedded hypersurface  $\Sigma_0$  in  $\mathbb{H}^{n+1}$  such that:

- (i)  $\partial\Sigma_0 = \Gamma$ ; and
- (ii)  $(\Sigma_n)_{n \in \mathbb{N}}$  converges to  $\Sigma_0$  locally uniformly.

Since  $\partial\Sigma_0 = \Gamma$  is smooth,  $T\Sigma_0$  is well defined at every point of  $\Gamma$ . We claim that at every point of  $\Gamma$ ,  $T\partial\Sigma_0$  makes an angle of  $\theta$  with  $\mathbb{R}^n$ , where:

$$\cos(\theta) = \tanh(d) = k.$$

Indeed, choose  $p_0 \in \Gamma$ , and let  $S \subseteq \mathbb{R}^n$  be a sphere contained in the complement of  $\Omega$  which is tangent to  $\Gamma$  at  $p_0$ . Let  $H$  be the totally geodesic hypersurface in  $\mathbb{H}^{n+1}$  such that  $\partial H = S$ . For all  $d > 0$ , let  $H_d \subseteq \mathbb{H}^{n+1}$  be the hypersurface lying above  $H$  at constant hyperbolic distance from  $H$ . By basic hyperbolic geometry and Axioms (ii) and (iii) of  $K$  (homogeneity and normality respectively), for all  $d$ :

$$K(H_d) = \tanh(d).$$

For all  $n$ , since  $\Sigma_n$  is a graph over  $\Omega$ , it only intersects  $H$  at  $p_0$ . Likewise, so does  $\Sigma_0$ . Thus, by the Geometric Maximum Principal, if  $\tanh(d) < k$ , then  $\Sigma_0$  only intersects  $H_d$  at  $p_0$ . We deduce that  $T\Sigma_0$  makes an angle of at most  $\theta$  with  $H$  at  $p_0$ , where:

$$\cos(\theta) = \tanh(d) = k.$$

Likewise, using spheres contained inside  $\Omega$  which are tangent to  $\Gamma$  at  $p_0$ , we show that the angle that  $T\Sigma_0$  makes with  $H$  at  $p_0$  is at least  $\theta$ , and this proves the assertion.

We now show that  $\Sigma_0$  is smooth away from  $\Gamma$ . Indeed, by Proposition 8.1, the set of points where  $\Sigma_0$  is strictly convex is open in  $\Sigma_0$ , and thus the set,  $X$ , of points where  $\Sigma_0$  is not strictly convex is closed. It follows from the basic properties of convex sets that  $X$  is stratified by totally geodesic components which are each the convex hulls of intersections of  $\Gamma$  with totally geodesic subspaces of  $\mathbb{H}^{n+1}$  (see [24] for details). Let  $X_0$  be one such component, and choose  $p_0 \in X \cap \Gamma$ . By definition of  $X_0$ ,  $T\Sigma_0$  makes a right angle with  $H$  at  $p_0$ , which is absurd. We conclude that  $X = \emptyset$ , and thus, by Proposition 8.1  $\Sigma_0$  is  $C^\infty$  away from the boundary.

Finally, observe that the supporting normal to  $\Sigma_0$  is continuous and uniquely defined along  $\Gamma$ . Thus, since the supporting normal to a convex set is continuous wherever it is uniquely defined, we now conclude moreover that  $\Sigma_0$  is  $C^1$ . This completes the proof.  $\square$

### Proposition 10.2

Let  $\Gamma$  and  $\Sigma$  be as in Proposition 10.1. If  $A$  is the shape operator of  $\Sigma$ , then, for all  $k \geq 0$  there exists  $B > 0$  such that, for all  $p \in \Sigma$ :

$$\|\nabla^k A(p)\| \leq B,$$

where  $\|\cdot\|$  and  $\nabla$  are the norm and covariant derivative over  $\Sigma$  induced by the hyperbolic metric.

**Proof:** Indeed, suppose the contrary. Identifying  $\mathbb{H}^{n+1}$  with the upper half space in  $\mathbb{R}^{n+1}$ , we denote by  $h : \mathbb{H}^{n+1} \rightarrow ]0, \infty[$  the  $(n+1)$ 'th component, which we view as a height function. Choose  $k \geq 0$  and let  $(p_n)_{n \in \mathbb{N}} \in \Sigma$  be such that:

$$(\|\nabla^k A(p_n)\|)_{n \in \mathbb{N}} \rightarrow +\infty.$$

By compactness, we may assume that  $p_n$  converges to a point  $p_0 \in \Gamma$ . For all  $n$ , let  $M_n$  be an isometry of  $\mathbb{H}^{n+1}$  which fixes  $p_0$  such that:

$$h(M_n(p_n)) = 1.$$

Bearing in mind that the angle that  $T\Sigma$  makes with  $\mathbb{R}^n$  along  $\Gamma$  is everywhere equal to  $\theta$ , where  $\cos(\theta) = k$ , we deduce that there exists a compact subset  $X$  of  $\mathbb{H}^{n+1}$  such that  $M_n(p_n) \in X$  for all  $n$ .

For all  $n$ , denote  $\Gamma'_n = M_n \Gamma_n$  and  $\Sigma'_n = M_n \Sigma_n$ . Since  $\Gamma$  is smooth,  $(\Gamma'_n)_{n \in \mathbb{N}}$  converges in the Hausdorff sense in  $\partial_\infty \mathbb{H}^{n+1}$  to a sphere. By the basic theory of convex sets,  $(\Sigma'_n)_{n \in \mathbb{N}}$  converges in the Hausdorff sense in  $\mathbb{H}^{n+1} \cup \partial_\infty \mathbb{H}^{n+1}$  towards a  $C^{0,1}$ , convex, embedded hypersurface,  $\Sigma'_0$  such that  $\partial \Sigma'_0 = \Gamma'_0$ .

As in the proof of Proposition 10.1,  $\Sigma'_0$  is smooth away from the boundary and has constant  $K$ -curvature equal to  $k$ . For all  $n$ , denote  $q_n = M_n p_n$ . We may assume that there exists  $q_0 \in \Sigma'_0$  towards which  $(q_n)_{n \in \mathbb{N}}$  converges. By Proposition 8.1,  $(\Sigma'_n)_{n \in \mathbb{N}}$  converges to  $\Sigma'_0$  in the  $C^\infty_{\text{loc}}$  sense near  $q_0$ . Consequently, there exists  $B > 0$  such that, for all  $n$ , if  $A'_n$  denotes the shape operator of  $\Sigma'_n$ , then:

$$\|\nabla^k A'(q_n)\| < B.$$

However, for all  $n$ ,  $M_n$  is an isometry, and so:

$$\|\nabla^k A(p_n)\| = \|\nabla^k A'(q_n)\| < B.$$

This is absurd by definition of  $(p_n)_{n \in \mathbb{N}}$ , and the result follows.  $\square$

This immediately yields:

### Corollary 10.3

For all  $k \geq 2$ , there exists  $B > 0$  such that if  $u : \Omega \rightarrow ]0, \infty[$  is the function of which  $\Sigma$  is the graph, then, throughout  $\Omega$ :

$$\|u^{k-1} D^k u\| < B.$$

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